nearly toric Schubert varieties and Dyck paths

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(joint with Mahir B. Can)

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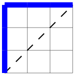
Sacred number 😇



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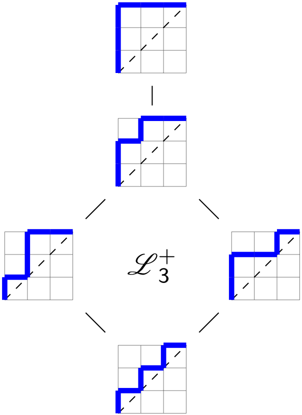

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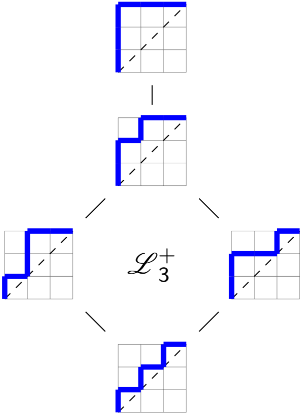

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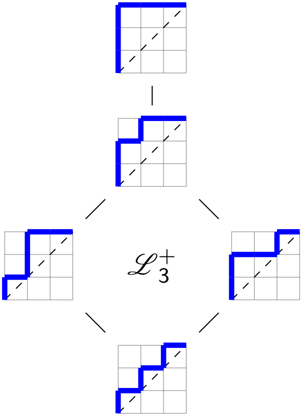

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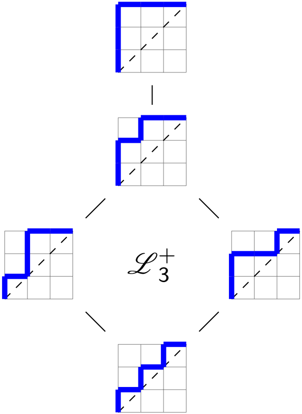

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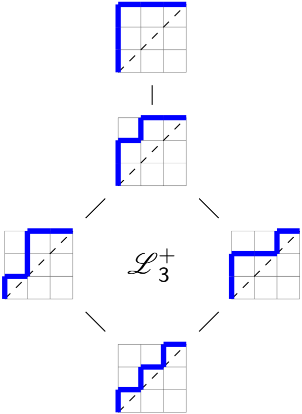

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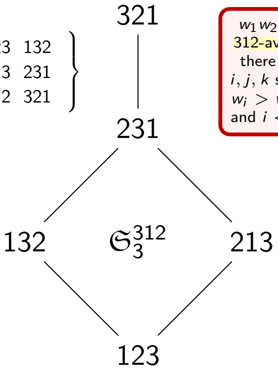

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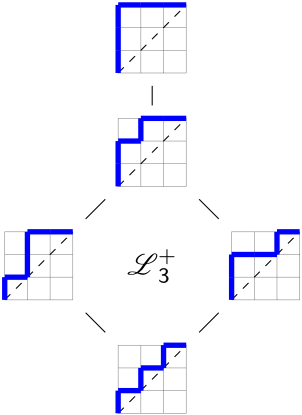

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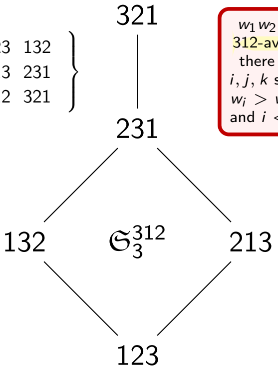

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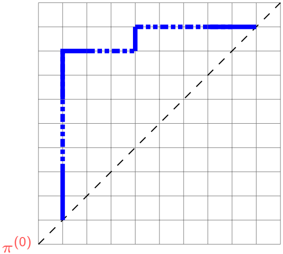
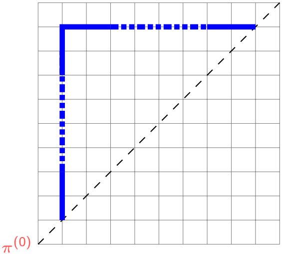
A Dyck path π is said to be spherical if every connected component of $\pi^{(0)}$ is either an elbow or a ledge, or every connected component of $\pi^{(1)}$ is an elbow, or a ledge whose E_+ extension is the initial step of a connected component of $\pi^{(0)}$.

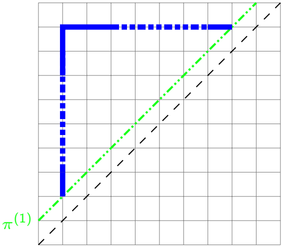
Spherical Dyck paths



Ledge



Elbow



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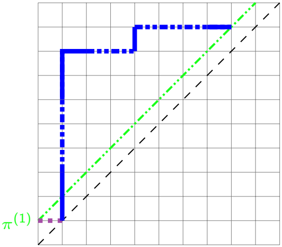

Ledge

Spherical varieties

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The **T**-complexity of **Y**, denoted by $c_{\mathbf{T}}(\mathbf{Y})$, is the codimension of the maximal torus **T** in **Y**. If the **T**-complexity of **T** is 1, we call **Y** a nearly toric variety.

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Ubiquitous examples:

- Toric varieties.
- Wonderful varieties.
- Reductive monoids.

Schubert varieties

The Ehresmann-Bruhat-Chevalley decomposition

$$\mathsf{GL}_n = \bigsqcup_{w \in \mathfrak{S}_n} \mathbf{B}\, w\, \mathbf{B}, \quad \mathfrak{S}_n \cong \mathbf{N}_{\mathsf{GL}_n}(\mathbf{T}) / \mathbf{T} \rightsquigarrow \mathsf{GL}_n(\mathbb{C}) / \mathbf{B} = \bigsqcup_{w \in \mathfrak{S}_n} \mathbf{B}\, w\, \mathbf{B} \,/\, \mathbf{B}$$

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The Schubert variety $X_{w\mathbf{B}}$ associated with w is the **B**-orbit closure

$$X_{w\mathbf{B}} := \overline{\mathbf{B}\, w\, \mathbf{B} / \mathbf{B}}.$$

- $X_{w\mathbf{B}}$ is normal.
- The irreducible representations of GL_n can be constructed via line bundles on $\mathrm{GL}_n / \mathbf{B}$.

Combinatorial gadget I

The $X_{v\,\mathbf{B}} \subseteq X_{w\,\mathbf{B}}$ induces a partial order on the **Weyl group**.

The Bruhat–Chevalley order on $\mathfrak{S}_n$ is defined by

$$v \leq w \iff X_{v\,\mathbf{B}} \subseteq X_{w\,\mathbf{B}}, \qquad \ell(w) = \dim X_{w\,\mathbf{B}}.$$

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If $w = s_{i_1} \cdots s_{i_\ell}$ and ℓ is minimal among all such expressions, then $\ell := \ell(w)$ is said to be the length of w, and the expression $s_{i_1} \cdots s_{i_\ell}$ is called a reduced decomposition for w

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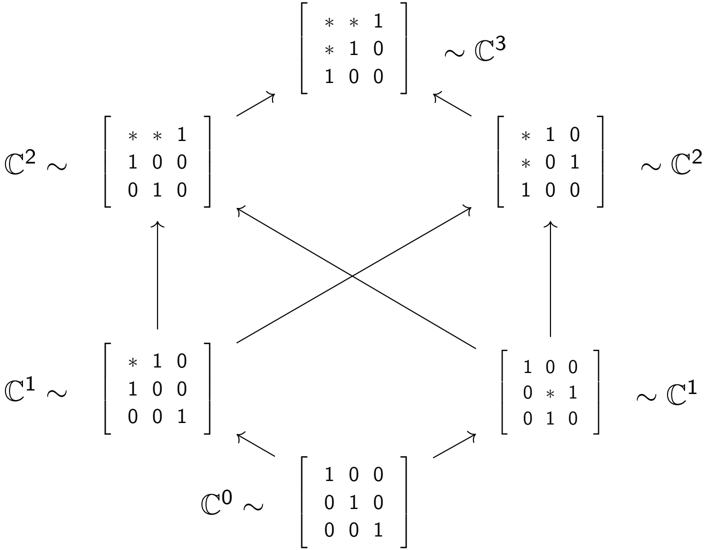
- $\mathrm{Red}(321) = \{s_1 s_2 s_1, s_2 s_1 s_2\}$

$v \leq w \iff$ a reduced decomposition of v is substring (subword) of some reduced decomposition for v.

- $1432 \leq 3412$ as $s_2 s_3 s_2$ is a subword of $s_2 s_1 s_3 s_2$.

Explanatory example



Combinatorial gadget II

For $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$ with $k \leq n$. The permutation w contains the pattern p if there exits a sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that $w(i_1) \cdots w(i_k)$ is in the same relative order as $p(1) \cdots p(k)$. If w does not contain p, then w is said to p-avoiding.

Combinatorial gadget II

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We call $X_{w\,\mathbf{B}}$ a partition Schubert variety (Ding's variety) if w is a 312-avoiding permutation.

- Let $\mathfrak{S}_n^{312}$ denote this family.
  - $w = 2\,3\,1\,8\,7\,6\,9\,5\,4\,10$ is in $\mathfrak{S}_{10}^{312}$.

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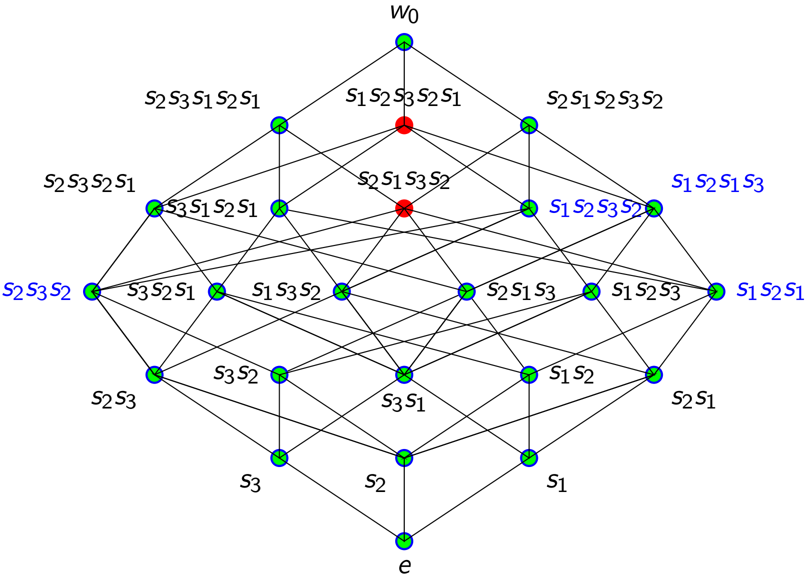
Theorem (Lakshmibai, Sandhya-1990)

The variety $X_{w\,\mathbf{B}}$ is smooth $\iff$ w avoids the patterns 3412 and 4231.

- $\mathfrak{S}_n^{312}$ is smooth.

Explanatory example



Combinatorial gadget: III

Let $\mathfrak{S}_{\mathbf{I}}$ be the parabolic subgroup of $\mathfrak{S}_n$ generated by $\mathbf{I} \subseteq \mathsf{S}$ and $w_0(\mathbf{I})$ its longest element.

- Denote $\mathsf{J}(w) := \{s \in \mathsf{S} : \ell(sw) < \ell(w)\}$ the left descent set of w
  - $w = 2\,3\,1\,8\,7\,6\,9\,5\,4\,10 \rightsquigarrow \mathsf{J}(w) = \{s_2, s_4, s_5, s_6, s_7\}$.
- A standard Coxeter element c in $\mathfrak{S}_{\mathbf{I}}$ is any product of the elements of $\mathbf{I}$ listed in some order.
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Let $\mathbf{L}$ denote the standard Levi factor of the parabolic subgroup of $X_{w\mathbf{B}}$ in GL_n. Let $\mathbf{B_L}$ be Borel subgroup of $\mathbf{L}$ containing $\mathbf{T}$. The Schubert variety $X_{w\mathbf{B}}$ is spherical if $\mathbf{B_L}$ has only finitely many orbits in $X_{w\mathbf{B}}$. In fact, $\mathbf{L_I}$ is given by $\mathsf{J}(w)$.

Classification

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Theorem (Gao, Hodges, Yong-2022)

$c_{\mathbf{B}_L}(X_{w\,\mathbf{B}}) = 0 \iff w_0(\mathrm{J}(w))w$ *is a standard Coxeter element (Boolean).*

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Theorem (Gaetz-2022)

$c_{\mathbf{B}_L}(X_{w\,\mathbf{B}}) = 0 \iff w$ *avoids the following 21 patterns*

$$\mathscr{P} := \left\{\begin{matrix} 24531 & 25314 & 25341 & 34512 & 34521 & 35412 & 35421 \\ 42531 & 45123 & 45213 & 45231 & 45312 & 52314 & 52341 \\ 53124 & 53142 & 53412 & 53421 & 54123 & 54213 & 54231 \end{matrix}\right\}$$

New characterization: ☕☕☕

Theorem (Can-D.)

Let w be in $\mathfrak{S}_n^{312}$. Let π denote the Dyck path of size n corresponding to w. Then $X_{w\mathbf{B}}$ is a spherical Schubert variety if and only if π is spherical.

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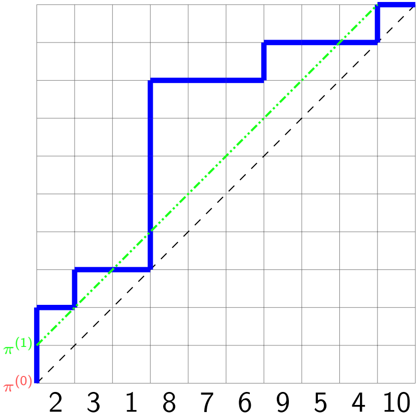

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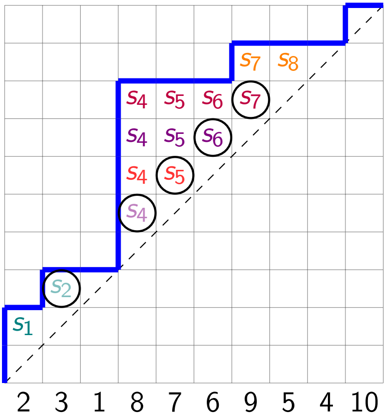

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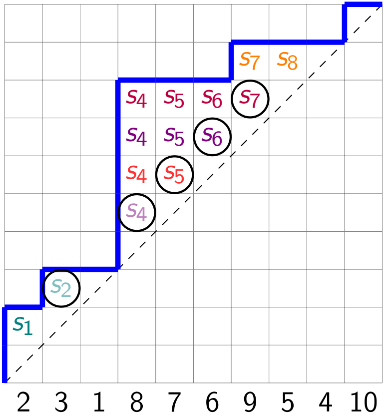
$$w^5 s_2 = 2\,1\,\mathbf{3}\,4\,5\,6\,7\,8\,9\,10 := w^6$$
$$w^6 s_1 = 1\,\mathbf{2}\,3\,4\,5\,6\,7\,8\,9\,10 := \mathrm{id}$$

$$\underbrace{\left(s_7 s_8 \,|\, s_4 s_5 s_6 s_7 \,|\, s_4 s_5 s_6 \,|\, s_4 s_5 \,|\, s_4 \,|\, s_2 \,|\, s_1\right)^{-1}}_{\mathrm{Red}(w)}$$



$$\mathfrak{S}_n^{312} \underset{\phi}{\overset{\psi}{\rightleftarrows}} \mathscr{L}_{n,n}^{+} \;;\; \ell(w) \longmapsto \mathrm{area}(\psi(w)) := \pi$$

Theorem (Lee, Masuda, Park-2021)

- $c_{\mathbf{T}}(X_{w\,\mathbf{B}}) = 1$ *and smooth* $\iff$ *w contains the pattern* 321 *exactly once and avoids* 3412 $\iff$ *there exists a reduced word of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions.*
- $c_{\mathbf{T}}(X_{w\,\mathbf{B}}) = 1$ *and singular* $\iff$ *w contains the pattern* 3412 *exactly once and avoids the pattern* 321.

Theorem (Lee, Masuda, Park-2021)

- $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ *and smooth* $\iff$ *w contains the pattern 321 exactly once and avoids 3412* $\iff$ *there exists a reduced word of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions.*
- $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ *and singular* $\iff$ *w contains the pattern 3412 exactly once and avoids the pattern 321.*

Corollary (Can-D.)

If $X_{w\mathbf{B}}$ is a partition Schubert variety of $\mathbf{T}$-complexity 1, then $X_{w\mathbf{B}}$ is nearly toric variety. In particular, the cardinality of this family is $2^{n-3}(n-2)$ for $n \geq 4$.

Theorem (Can-D.)

Let $X_{w\mathbf{B}}$ be a singular Schubert variety of $\mathbf{T}$-complexity 1. Then $X_{w\mathbf{B}}$ is nearly toric variety. Furthermore, let b_n be the cardinality of this family. Then the generating series of b_n is given by A001871 in the OEIS.

Shortcoming and upcoming work.

(i) If $w = 25314$, we found out that $c_{\mathbf{T}}(X_{w\,\mathbf{B}}) = 1$ is smooth, yet $c_{\mathbf{B_L}}(X_{w\,\mathbf{B}}) \neq 0$. By using SAGEMATH, we discovered that

n	1	2	3	4	5	6	7	8	9
r_n	0	0	1	6	24	84	275	864	2639

According to OEIS, this sequence is given by $r_{n+2} = n \cdot \mathscr{F}_{2n}$ where $\mathscr{F}_m$ is the m-th Fibonacci number.

(ii) By inspection, the cardinality of $\{w \in \mathfrak{S}_n^{312} : c_{\mathbf{B_L}}(X_{w\,\mathbf{B}}) = 0\}$ grows as

n	1	2	3	4	5	6	7	8	9
t_n	1	2	5	14	39	107	291	789	2138

We are still figuring out the generating series $\sum_{n\geq 3} r_n x^n, \quad \sum_{n\geq 1} t_n x^n \ldots$

Counting spherical Dyck paths

Theorem (Bankston-D)

The number of spherical Dyck paths $\mathscr{S}_n$ is given by

$$|\mathscr{S}_n| = \begin{cases} 1 & n = 1 \\ \sum_{k=2}^{n-1} |\mathscr{S}_{n-k}|\pi_k^{(1)} + \pi_n^{(1)} + |\mathscr{S}_{n-1}| & n \geq 2 \end{cases}.$$

where

$$\pi_n^{(1)} = \begin{cases} 1 & 1 \leq n \leq 2 \\ 3 \cdot 2^{n-3} - 1 & n \geq 3 \end{cases}.$$

Surprisingly, $\pi_n^{(1)}$ counts the independence number of n-Mylcielski graph based on A266550-OIES.

Thank You/Gracias/Obrigado 😄

https://arxiv.org/abs/2212.01234

"Stones on the road? I save every single one, and one day I'll build a castle." *Fernando Pessoa.*