A naive introduction to Affine Schemes

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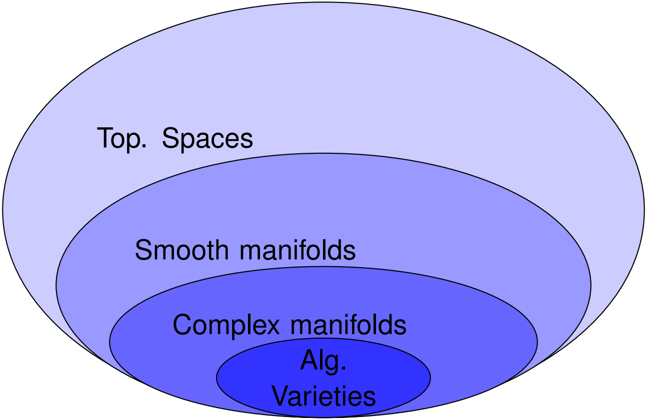
Outline

Figure: Drawn by my sister.



"The introduction of the cipher 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps..." A. Grothendieck

- A priori, we are interested in geometric properties of the set of zeroes

$$\mathbf{V}(f_1, ...f_r) := \{(t_1, ..., t_n) \in \mathbf{k}^n : f_i(t_1, ..., t_n) = 0 \quad \forall i\} \subset \mathbf{k}^n . \tag{1}$$

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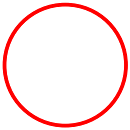
for some polynomials $f_1, ..., f_r \in \mathbf{k}[T_1, ..., T_n]$. In other words, there is a triple

$$\left(\underbrace{\mathbf{k}}_{\text{field}}, \overbrace{T}^{\text{unknowns}}, \underbrace{F}_{\text{polynomials}} \right)$$

so that these objects interplay as in (1)

e. g. If $f = T_2^2 + T_1^2 - 1 \in \mathbf{k}[T_1, T_2]$, we see



$\mathbf{V}(f)$

- Let $M \subset \mathbf{k}[T_1, ..., T_n] = \mathbf{k}[\underline{T}]$ be a subset. The set of common zeroes of the polynomials in M is denoted by

$$\mathbf{V}(M) := \{(t \in \mathbf{k}^n : f(t) = 0 \quad \forall f \in M\}.$$

  - Since $\mathbf{k}$ is a field, $\exists$ finitely many elements $f_1, ..., f_r \in M$ for every subset $M \subset \mathbf{k}[\underline{T}]$ such that $\mathbf{V}(M) = \mathbf{V}(f_1, ..., f_r)$.

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- The sets $\mathbf{V}(\mathfrak{a})$, where $\mathfrak{a}$ runs through the set of ideals of $\mathbf{k}[\underline{T}]$, are the closed sets of a topology on $\mathbf{k}^n$, called the Zariski topology.

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- itself.
- One point $\{x\} = \mathbf{V}(\mathfrak{a}_x)$.
- $\mathbf{V}(f)$, f is irreducible poly.

- What if we pick any set W from $\mathbb{A}^n_{\mathbf{k}}$, is there any *gadget* from $\mathbf{k}[\underline{T}]$ such that $\mathbf{V}(\text{gadget}) = W$?

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- Let $W \subset \mathbb{A}_n$ be any subset.

$$\mathbf{I}(W) := \{f \in \mathbf{k}[\underline{T}] : f(x) = 0 \quad \forall x \in X\}.$$

Theorem (Hilbert's Nullstellensatz)

- *For any affine alg. set $W \subset \mathbb{A}^n_{\mathbf{k}}$, we have $\mathbf{V}(\mathbf{I}(W)) = W$.*
- *For any ideal $\mathfrak{a} \subset \mathbf{k}[\underline{T}]$ we have $\mathbf{I}(\mathbf{V}(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*
  - *The radical of $\mathfrak{a} \subset \mathbf{k}[\underline{T}]$ is $\sqrt{\mathfrak{a}} := \{f \in \mathbf{k}[\underline{T}] : \exists r \in \mathbb{Z}_{\geq 0} \quad f^r \in \mathfrak{a}\}$.*

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$$\begin{array}{ccc} \{\text{affine alg. sets in } \mathbb{A}^n_{\mathbf{k}}\} & \overset{1:1}{\longleftrightarrow} & \{\text{radicals ideals in } \mathbf{k}[\underline{T}]\} \\ W & \mapsto & \mathbf{I}(W) \\ \mathbf{V}(\mathfrak{a}) & \leftarrow\!\shortmid & \mathfrak{a} \end{array}$$

- A topological space $X \neq \emptyset$ is called irreducible whether X can't be expressed as the union of two proper closed subsets.
- A topological space X is called Noetherian if every descending chain

$$X \supseteq Z_1 \supseteq Z_2 \supseteq \cdots$$

  of closed subsets of X becomes stationary.
- Let $X \subset \mathbb{A}^n_{\mathbf{k}}$ be any subset. Then X is Noetherian.
- Any alg. set W is irreducible if and only if $A(W)$ is an integral domain.
  - The coordinate ring is defined by

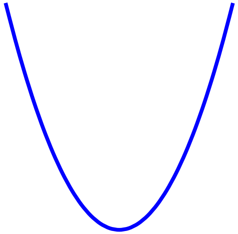
$$A(W) := \mathbf{k}[\underline{T}]/\,\mathbf{I}(W)$$

e. g. Let $W = \mathbf{V}(T_2 - T_1^2) \subset \mathbb{A}^2_{\mathbb{C}}$ be affine alg. set.

- We parse

$$A(W) = \mathbb{C}[T_1, T_2]/\,\mathbf{I}(W) \cong \mathbb{C}[T_1, T_2]/(T_2 - T_1^2) \cong \mathbb{C}[T_1].$$

- Can we talk about dimension of W?



- Given two alg. sets $X \subset \mathbb{A}^m_{\mathbf{k}}$ and $Z \subset \mathbb{A}^n_{\mathbf{k}}$. A morphism $X \to Z$ is map $f : X \to Z$ of the underlying sets such that there exist poly. $f_1, ..., f_n \in \mathbf{k}[T_1, ..., T_m]$ with $f(x) = (f_1(x), ..., f_n(x))$ for all $x \in X$.

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  - Morphism of affine alg. sets are continuous.
  - Composition of morphism is again a morphism of alg. sets.
- **e. g.**
  - $\mathbb{A}^1 \to \mathbf{V}(T_1 - T_2^2); \quad x \mapsto (x^2, x)$ is a morphism. Moreover, its inverse $(x, y) \mapsto y$ is a morphism.

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  - $\exp : \mathbb{A}^1_{\mathbb{C}} \to \mathbb{A}^1_{\mathbb{C}}$ is not a morphism of alg. sets.

- The set $\mathrm{Hom}(X, \mathbb{A}^1_{\mathbf{k}})$ carries the structure of a $\mathbf{k}$-algebra with addition and multiplication

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- $\mathbf{V}(\mathfrak{a}) = \{x \in X \; f(x) = 0 \quad \forall f \in \mathfrak{a}\} = \mathbf{V}(\pi^{-1}(\mathfrak{a})) \cap X$, $\mathfrak{a} \subset \Gamma(X)$.
  - For $f \in \Gamma(X)$, we set $D(f) := \{x \in X : f(x) \neq 0\} = X \setminus \mathbf{V}(f)$.
- The principal open sets $D(f)$ form a basis of the Zariski topology.

In short,

$$\left\{\begin{matrix}\text{Affine algebraic}\\ \text{sets } X \text{ over}\\ \text{field } \mathbf{k}\end{matrix}\right\} \Longleftrightarrow \left\{\begin{matrix}\text{Commutative rings:}\\ \text{Algebra over } \mathbf{k}\\ \text{Finitely generated}\\ \text{No nilpotents}\end{matrix}\right\}$$

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- $\mathbf{k}[T_1]$ behaves pretty similar to $\mathbb{Z}$ or $\mathbb{Z}[i]$...
- $\mathbf{k}[T_1] \subset k(T_1)$ and $\mathbf{k}[T_1]_{(T_1)} = \frac{p}{q}$ though...
- There is not distinction between $\mathbf{V}(T_1) \cap \mathbf{V}(T_2) \subset \mathbb{A}^2_{\mathbf{k}}$ and $\mathbf{V}(T_2) \cap \mathbf{V}(T_1^2 - T_1) \subset \mathbb{A}^2_{\mathbf{k}}$ although...

Thus (dream),

$$\text{Affine Schemes} \Longleftrightarrow \text{Commutative rings}$$

Thank You/Gracias!

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