Q_p Space on Riemann Surfaces

Néstor Fernando Díaz Morera
Instituto Politécnico Nacional, ESFM, México

I encuentro SCM-SMM, Universidad del Norte
Barranquilla

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Definition

Let (X, τ) be a topological space. We way a function $u : X \to [-\infty, \infty)$ is *upper semicontinuous (u.s.c)* if the set $\{x \in X \mid u(x) < \alpha\}$ belongs to τ for each $\alpha \in \mathbb{R}$. In other words, $u^{-1}([-\infty, \alpha))$ is open in X.

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Definition

Let $U \subset \mathbb{C}$ open. A function $u : U \to [-\infty, \infty)$ is called *subharmonic* if it is *upper semicontinuous* and satisfies the *local submean inequality*, i.e. given $w \in U$, there exists $\rho > 0$ such that

$$u(w) \leq \frac{1}{2\pi} \int_0^{2\pi} u(w + re^{it})dt \quad \text{for } r \in [0, \rho).$$

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Example

If f is holomorphic on $U \subset \mathbb{C}$ open $\Rightarrow \log |f|$ is subharmonic on U.

Proposition

Let u, v be subharmonic function on an open $U \subset \mathbb{C}$, then

(i) $\max(u, v)$ *is subharmonic on U.*

(ii) $\alpha u + \beta v$ *is subharmonic on U for all $\alpha, \beta \geq 0$.*

Theorem (Maximum Principle)

Let u be a subharmonic function on a domain $G \subset \mathbb{C}$.

(i) *If u attains a global maximum on $G \Rightarrow u \equiv C$ for some constant C.*

(ii) *If $\lim_{z \to \zeta} u(z) \leq 0$ for all $\zeta \in \partial G \Rightarrow u \leq 0$ on G.*

Criteria for Subharmonicity

Theorem

Let U be an open subset of $\mathbb{C}$ and let $u : U \to [-\infty, \infty)$ be an upper semicontinuous function. Then the following are equivalent.

(i) *The function u is subharmonic on U.*

(ii) *Whenever $\overline{\Delta}(w, \rho) \subset U$, then for $r < \rho$ and $t \in [0, 2\pi)$*

$$u(w + re^{it}) \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{\rho^2 - r^2}{\rho^2 - 2\rho r \cos(\theta - t) + r^2} \phi(w + \rho e^{i\theta}) d\theta.$$

(iii) *(Harmonic Majoration) Whenever D is precompact subdomain of U and h is harmonic function on D satisfying*

$$\lim_{z \to \zeta} \sup (u - h)(z) \leq 0 \; for \; \zeta \in \partial D,$$

then $u \leq h$ on D.

Definition

Let $\gamma : I \to \mathbb{D}$ be a smooth curve. The *length* is to be

$$L_\rho(\gamma) = \int_\gamma \left(\frac{2}{1-|z|^2}\right)|dz| = \int_0^1 \left(\frac{2}{1-|\gamma(t)|^2}\right)|\gamma'(t)|dt.$$

Definition

Let

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be the *density of the hyperbolic distance* in $\mathbb{D}$.

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Definition

Let

$$\lambda_{\mathbb{D}}(z) = \frac{1}{1-|z|^2}$$

be the *density of the hyperbolic distance* in $\mathbb{D}$. The *hyperbolic distance* between two points z_0 and z_1 in $\mathbb{D}$ is given by

$$\rho_{\mathbb{D}}(z_0, z_1) := \inf\left\{\int_\gamma \lambda_{\mathbb{D}}(z)|dz|\right\}.$$

Definition

Let G be a domain in the extended plane $\widehat{\mathbb{C}}$ and let $a \in \Omega$. A *Green's function of G with singularity at a* is a function $g : \Omega \setminus \{a\} \to \mathbb{R}$ which holds

(i) $g(z, a)$ is harmonic in $\Omega \setminus \{a\}$.

(ii) $G(z) = g(z, a) + \log |z - a|$ is harmonic is a disk about a.

(iii) $\lim_{z \to w} g(z, a) = 0$ for each $w \in \partial\Omega$.

Proposition (Example)

Let $\varphi_a : \mathbb{D} \to \mathbb{D}; \quad \varphi_a(z) \mapsto \frac{a-z}{1-\bar{a}z}, \quad a \in \mathbb{D}$ *be a Möbius transformation, then*

$$g(z, a) = \log \frac{1}{|\varphi_a(z)|}$$

is the Green's function for $\mathbb{D}$ *with singularity at* $z = a$.

Definition

A topological surface M is a Hausdorff topological space provided with collection $\{\varphi_i : U_i \to \varphi_i(U_i)\}$ of homeomorphisms (called charts) from open subsets $U_i \subset M$ (called coordinated neighbourhoods) to open subsets $\varphi_i(U) \subset \mathbb{C}$ such that:

(i) $M = \bigcup_{i \in I} U_i$.

(ii) Whenever $U_i \cup U_j \neq \emptyset$, the transition functions

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is a homeomorphism.

A collection of charts fulfilling these properties is called a (topological) atlas, and the inverse φ_i^{-1} is called a parametrization.

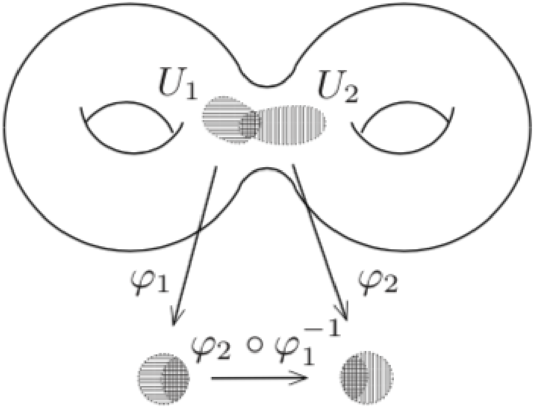

Figure: The transition function between two coordinates charts.

Definition

A Riemann surface R is a connected topological surface such the transition functions of the atlas are holomorphic mappings between open subsets of the complex plane $\mathbb{C}$ i.e, it is pair (R, Σ).

Example

Let $M = \mathbb{C}$, and let U be any open subset. Define $\varphi_U(x, y) = x + iy$ from (considered as a subject of $\mathbb{C}$) to the complex plane. This is a complex chart on $\mathbb{C}$. Moreover Let M be $\mathbb{C}$ itself, considered topologically as $\mathbb{R}^2$. Therefore, it is a Riemann surface which is called *complex plane*.

Example

Let $\mathbb{S}^2 = \{(x, y, t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 = 1\}$ be denoted the unit 2-sphere. Put $t = 0$ plane as a copy of the complex plane $\mathbb{C}$, with $(x, y, 0)$ being identified with $z = x + iy$. Let's us considere the following two charts

$$U_1 = \mathbb{S}^2 \setminus \{(0,0,1)\}, \quad \varphi_1(x,y,t) = \frac{x}{1-t} + i\frac{y}{1-t}$$
$$U_2 = \mathbb{S}^2 \setminus \{(0,0,-1)\}, \quad \varphi_2(x,y,t) = \frac{x}{1+t} - i\frac{y}{1+t}$$

Since $\dfrac{x-iy}{1+t} = \dfrac{1-t}{x+iy}$, it follows that the transition function is

$$\varphi_2 \circ \varphi_1^{-1}(z) = \frac{1}{z}$$

which is holomorphic on a domain $\varphi_1(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$.

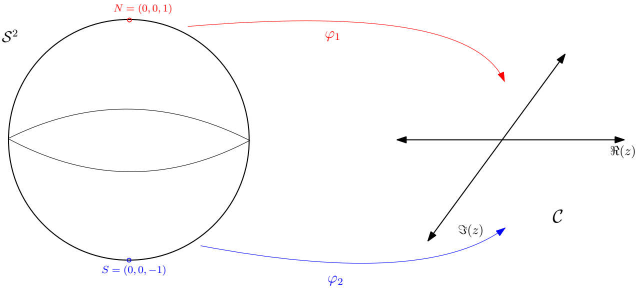

Figure: Compatible charts on $\mathbb{S}^2$.

Definition

Let M be a Riemann surface and $Y \subset M$ a open subset. A function $f : Y \to \mathbb{C}$ is called **holomorphic**, if for every chart $\psi : U \to V$ on M the function

$$f \circ \psi^{-1} : \psi(U \cap Y) \to \mathbb{C}$$

is holomorphic in the usual sense on the open set $\psi(U \cap Y) \subset \mathbb{C}$.

Definition

Suppose M and N are Riemann surfaces. A continuous map $F : M \to N$ is called *holomorphic*, if for every pair of charts $\psi_1 : U_1 \to V_1$ on M and $\psi_2 : U_2 \to V_2$ on N with $f(U_1) \subset U_2$, the mapping

$$\psi_2 \circ F \circ \psi_1^{-1} : V_1 \to V_2$$

is holomorphic in the usual sense.

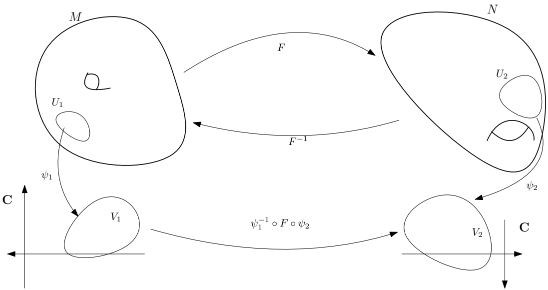

Figure: Morphism between Riemann surfaces.

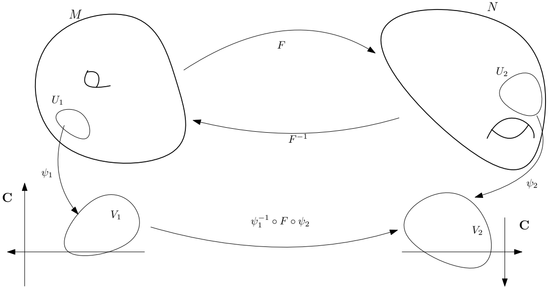

Figure: Morphism between Riemann surfaces.

Definition

A function $F : M \to N$ is said to be a **biholormorphic** if it is a bijective and both $F : M \to N$ and $F^{-1} : N \to M$ are holomorphic.

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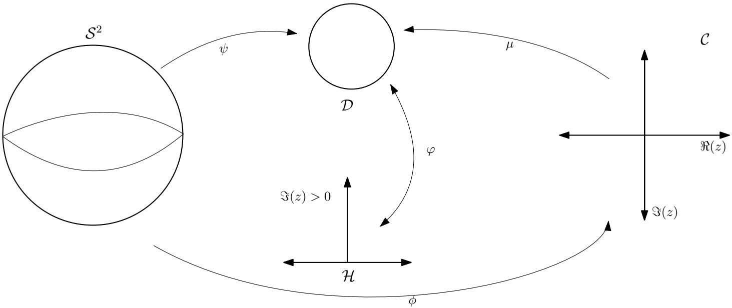

Figure: Likely biholomorphisms among $\mathbb{S}^2$, $\mathbb{D}$ and $\mathbb{C}$.

Theorem (Riemann Mapping Theorem)

Any non-empty simply connected domain $\Omega \subset \mathbb{C}$, which is not $\mathbb{C}$, is ***biholomorphic*** *to the unit disc $\mathbb{D}$.*

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Any non-empty simply connected domain $\Omega \subset \mathbb{C}$, which is not $\mathbb{C}$, is ***biholomorphic*** *to the unit disc $\mathbb{D}$.*

They aren't biholomorphic among them since:

- $\mu : \mathbb{C} \to \mathbb{D}$ neither by Liouville's theorem.
- $\psi : \mathbb{S}^2 \to \mathbb{D}$ and $\phi : \mathbb{S}^2 \to \mathbb{C}$ neither by compactness of $\mathbb{S}^2$.

However, $\mathbb{H}$ and $\mathbb{D}$ are biholomorphic via the following Möbius transformation

$$\varphi(z) = \frac{z - i}{z + i}$$

Are there other Riemann surfaces beside $\mathbb{C}$, $\mathbb{D}$ and $\widehat{\mathbb{C}}$?

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Theorem (The Uniformization Theorem (Poincaré, Koebe -1907))

Every simply connected Riemann surface M is biholomorphic either to

- $\mathbb{D}$ *(hyperbolic),*
- $\mathbb{C}$ *(parabolic),*
- $\widehat{\mathbb{C}}$ *(elliptic).*

Definition

Let E and X be topological surfaces. A continuous mapping $\pi : E \to X$ is a *covering map* if the following holds.

(i) Every point $x \in X$ has a open neighborhood U such that its preimage $\pi^{-1}(U)$ can be represented as

$$\pi^{-1}(U) = \bigsqcup_{j \in J} V_j.$$

where the $\{V_j\}_{j \in J}$ are disjoint open subsets of E.

(ii) In particaular, π is a local homeomorphism.

Example

Let $X = \mathbb{S}^1$ and $E = \mathbb{R}$ be the circle and the real line respectively. Then the mapping $p(t) = e^{2\pi it}$ is a covering.

Proposition

If X has a holomorphic structure, then E inherits a unique Riemann surface structure such that π is holomorphic.

Definition

Let $\pi : E \to R$ be a covering map. It is called a universal covering of a topological space E if E is simply connected.

Theorem

The universal covering for any Riemann surface R is either $\mathbb{C}$, $\mathbb{D}$ or $\mathbb{S}^2$

Theorem

Every Riemann surface R is biholomorphic to a quotient $\tilde{S}/\Gamma$, where $\tilde{S}$ is $\mathbb{D}, \mathbb{C}, \widehat{\mathbb{C}}$ and Γ is a group of automorphism of $\tilde{S}$ which acts freely and properly discontinuously.

Definition

A function f is called a **Bloch function** if it is holomorphic on $\mathbb{D}$ and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < +\infty. \tag{1}$$

We will denote $\mathcal{B}$ the family of all Bloch functions.
It's called a **little Bloch** function if

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Proposition

- *Every bounded function $f:\mathbb{D}\to\mathbb{C}$ is a Bloch function.*
- *If $f\in\mathcal{B}$, then for all $z\in\mathbb{D}$*

$$|f(z)| \leq |f(0)| + M|\lambda(|z|)|$$

where $\lambda(z) = \log(1-z)$ *and* $M = \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$.

Proposition

- *The set* $\mathcal{B}$ *equipped with the norm*

$$\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|$$

  is a Banach space.
- $\mathcal{B}$ *is not separable.*
- *Let* $f \in \mathcal{B}$ *and* $\varphi : \mathbb{D} \to \mathbb{D}$ *be a conformal mapping of* $\mathbb{D}$ *onto itself, then* $h(z) = f(\varphi(z)) \in \mathcal{B}$.

Definition

A Dirichlet space $\mathcal{D}$ is the collection of all holomorphic function on $\mathbb{D}$ such that

$$\iint_{\mathbb{D}} |f'(z)|^2 dxdy < \infty \text{ where } z = x + iy.$$

It can equipped with the following norm:

$$\|f\|_{\mathcal{D}} = \left(|f(0)|^2 + \iint_{\mathbb{D}} |f'(z)|^2 dxdy \right)^{1/2}.$$

Proposition

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is holomorphic on* $\mathbb{D}$, *then*

$$\frac{1}{\pi}\iint_{\mathbb{D}} |f'(z)|^2 dxdy = \sum_{n=1}^{\infty} n|a_n|^2.$$

Proposition

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is holomorphic on $\mathbb{D}$, then

$$\frac{1}{\pi}\iint_{\mathbb{D}} |f'(z)|^2 dxdy = \sum_{n=1}^{\infty} n|a_n|^2.$$

Observations

A briefly properties about $\mathcal{D}$ spaces are mentioned in the following.

- *The $\mathcal{D}$ space is conformally equivalent.*
- *The $\mathcal{D}$ space is a Banach space with the norm $\|\cdot\|_{\mathcal{D}}$.*

Definition

For $p > 0$, let $\mathcal{Q}_p$ denote the space of all holomorphic function satisfying

$$\sup_{\alpha \in \mathbb{D}} \iint_{\mathbb{D}} |f'(z)|^2 g(z,\alpha)^p dxdy < \infty, \tag{3}$$

where $g(z,\alpha)$ is the Green's function with a logarithm singularity at α.

Definition

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Definition

For $p > 0$, let $\mathcal{Q}_{p,0}$ denote the space of all holomorphic function satisfying

$$\lim_{|\alpha| \to 1} \iint_{\mathbb{D}} |f'(z)|^2 g(z,\alpha)^p \, dxdy = 0, \tag{4}$$

where $g(z,\alpha)$ is the Green's function with a logarithm singularity at α.

Observations

We will view a briefly glance about $\mathcal{Q}_p$ and $\mathcal{Q}_{p,0}$. In fact, how they relate with the previous spaces.

- *For $p = 0$, it follows $\mathcal{Q}_p = \mathcal{D}$.*
- *For $p > 1$, it follows $\mathcal{Q}_p = \mathcal{B}$.*
- *For $p > 1$, $\mathcal{Q}_{p,0} = \mathcal{B}_0$.*

Theorem

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function, then the following conditions are equivalent

(i) $f \in \mathcal{B}$.

(ii) $\{f\}_{\mathcal{Q}_p} < \infty$ *for all* $p > 1$.

(iii) $\{f\}_{\mathcal{Q}_p} < \infty$ *for some* $p > 1$.

Theorem

For $0 < p < q < \infty$, it holds $\mathcal{Q}_p \subset \mathcal{Q}_q$.

Definition

Let R be a Riemann surface. A real-valued function $h : R \to \mathbb{R}$ is *harmonic* at p belonging to R if there exists a coordinate disk (Δ, φ) containing p such that $h \circ \varphi^{-1} : \mathbb{D} \to \mathbb{R}$ is harmonic function. If h is harmonic at each $p \in R$, we say h is a **harmonic function** on R.

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Definition

A continuous function $u : R \to [-\infty, \infty)$ is **subharmonic** if for every coordinate disk (Δ, φ) and $h : \overline{\Delta} \to \mathbb{R}$ is a harmonic function such that $u(p) \leq h(p)$ for all $p \in \partial\Delta$, then $u(p) \leq h(p)$ for all $p \in \Delta$.

Theorem (Maximum Principle)

Let u be a subharmonic function on a Riemann surface R. If u attains a maximum at $p \in R$, then u is a constant function.

Observations

Let (Δ, φ) be a coordinate disk and u be a subharmonic function on R. By using the Poisson integral, we can solve the Dirichlet problem

$$\begin{cases} \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 & in\ \mathbb{D} \\ w = u \circ \varphi^{-1} & on\ \partial\mathbb{D}. \end{cases}$$

Let $u_\Delta : R \to \mathbb{R}$ by

$$u_\Delta(p) = \begin{cases} u(p) & if\ p \notin \Delta \\ (w \circ \varphi)(p) & if\ p \in \Delta. \end{cases}$$

Then u_Δ is continuous on R and harmonic on Δ.

Definition

A **Perron family** on R is a collection $\mathcal{F}$ of subharmonic functions such that

(i) If $u_1, u_2 \in \mathcal{F}$, then $\max\{u_1, u_2\} \in \mathcal{F}$.

(ii) If $u \in \mathcal{F}$, then $u_\Delta \in \mathcal{F}$.

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Theorem

Let $\mathcal{F}$ be a Perron family on R. Then $u(p) = \sup\{v(p) \mid v \in \mathcal{F}\}$ is either harmonic or $u(p) = +\infty$ for all $p \in R$.

Proposition

Fix a point $q \in R$ and let (Δ, φ) be a coordinate disk containing q such that $\varphi(q) = 0$. Let $\mathcal{P}$ be a family of subharmonic functions on $R \setminus \{q\}$ such that

(i) *Every $u \in \mathcal{P}_q$ has compact support.*

(ii) *Every $u \in \mathcal{P}_q$is such that $v(p) = u(p) + \log|\varphi(p)|$ is subharmonic on Δ.*

Then, $\mathcal{P}_q$ is a Perron family on $R \setminus \{q\}$.

Definition

Suppose $\sup\{u(p) : u \in \mathcal{P}_q\} < \infty$ for some $p \in R$. A **Green's function** for R with singularity at q is defined as $g(p, q) = \sup\{u(p) \mid u \in \mathcal{P}_q\}$ for all $p \in R \setminus \{q\}$.

Proposition

Let $g(p,q)$ be a Green's function for R with a singularity at q. Then

(i) $g(p,q) > 0$.

(ii) $g(p,q)$ *is harmonic for all* $p \in R \setminus \{q\}$.

(iii) *If* (Δ, φ) *is a coordinate disk such that* $\varphi(q) = 0$, *then* $h(p) = g(p,q) + \log|\varphi(p)|$ *is harmonic on* Δ.

Proposition

Let $g(p,q)$ be a Green's function on R with a pole at q. Then

$$\inf_{p \in R} g(p,q) = 0.$$

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Definition

Let R be a Riemann surface. Then, R is **hyperbolic** if it admits a Green's function.

Definition

Let $0 < p < \infty$ and R be a hyperbolic Riemann surface. We say that a holomorphic function $f : R \to \mathbb{C}$ belongs to $\mathcal{Q}_p(R)$ if

$$\|f\|^2_{\mathcal{Q}_p(R)} = \sup_{z_0 \in R} \iint_R |f'(z)|^2 (g_R(z, z_0))^p dz \wedge d\bar{z}, \tag{5}$$

.

Observations

In fact, let $R = \mathbb{D}$ be, we get the Bloch space. Moreover, we denote $\mathcal{Q}_0(R)$ and $\mathcal{Q}_1(R)$ by $\mathcal{D}(R)$ and $\mathcal{BMOA}(R)$ as the Dirichlet and BMOA spaces on R, respectively.

Remark

Let L be the length of a closed curve and the area A of the planar region on $\mathbb{R}^2$ that it encloses, then

$$4\pi A \leq L^2.$$

Proposition

Let R be a Riemann surface, $\Omega \subset R$ a precompact domain and $\Gamma = \partial\Omega$ piecewise smooth boundary. If $f : R \to \mathbb{C}$ is holomorphic, then the following isoperimetric inequality holds:

$$4\pi|f(\Omega)| \leq |f(\Gamma)|^2$$

where $|f(\Omega)|$ and $|f(\Gamma)|$ denote the area of $f(\Omega)$ as covering surface and the length of $f(\Gamma)$ respectively.

Theorem

Let R be a hyperbolic Riemann surface and $g_R(z, z_0)$ be its Green's function with singularity at z_0. For $t \geq 0$, let $R_t = \{z \in R \mid g_R(z, z_0) > t\}$. If $f : R \to \mathbb{C}$ is holomorphic, then the function

$$A(t) = \iint_{R_t} |f'(z)|^2 dz \wedge d\bar{z}$$

has the following three properties:

(i) *$A(t)$ is continuous and decreasing with increasing $t \geq 0$.*

(ii) *$e^{2s} A(s) \leq e^{2t} A(t)$ for $s \geq t \geq 0$.*

(iii) *For $p \geq 0$ and $t \geq 0$,*

$$\frac{i}{2} \iint_{R_t} |f'(z)|^2 (g_R(z, z_0))^p dz \wedge d\bar{z} = \int_0^{\infty} A(s) ds^p = -\int_t^{\infty} s^p dA(S).$$

The right-side integral will be understood under Riemann-Stieltjes integration.

Proposition

Given a nonnegative function $A(t)$ on $(0, \infty)$ with the following two properties:

(i) *$A(t)$ is continuous and decreasing with increasing $t > 0$.*

(ii) *$e^{2t_2}A(t_2) \leq e^{2t_1}A(t_1)$ when $t_2 \geq t_1 > 0$.*

For $p, t \geq 0$, let $B_p(t) = -\int_t^\infty s^p dA(s)$. If $p \geq q \geq 0$, then

$$B_p(0) \leq \frac{2^q\Gamma(p+1)}{2^p\Gamma(q+1)} B_q(0).$$

Furthermore,

$$B_p(0) = \frac{2^q\Gamma(p+1)}{2^p\Gamma(q+1)} B_q(0) < \infty$$

if and only if

$$A(0) = \lim_{t \to 0} A(t) < \infty \quad and \quad A(t) = e^{-2t}A(0), \quad t > 0.$$

Theorem

Let $0 \leq q < p$ and R be a hyperbolic Riemann surface with $w \in R$. Then,

(i) *For any holomorphic $f : R \to \mathbb{C}$,*

$$\iint_R |f'(z)|^2 (g_R(z,w))^p dz \wedge d\bar{z} \leq \left(\frac{2^q \Gamma(p+1)}{2^p \Gamma(q+1)}\right)^{1/2} \iint_R |f'(z)|^2 (g_R(z,w))^q dz \wedge d\bar{z}$$

(ii) $\mathcal{Q}_q(R) \subset \mathcal{Q}_p(R)$ *with*

$$\|f\|^2_{\mathcal{Q}_p(R)} \leq \left(\frac{2^q \Gamma(p+1)}{2^p \Gamma(q+1)}\right) \|f\|^2_{\mathcal{Q}_q(R)}, \quad f \in \mathcal{Q}_q(R).$$

Definition

Let $p : \mathbb{D} \to R$ be the universal covering mapping of a Riemann surface R and suppose $w_0, w_1 \in R$. We define the *hyperbolic distance* between w_0 and w_1 on R by

$$\rho_R(w_0, w_1) := \inf\{\rho_{\mathbb{D}}(z_0, z_1) \mid p(z_0) = w_0 \quad \text{and} \quad p(z_1) = w_1\},$$

where $\rho_{\mathbb{D}}(z_0, z_1)$ is defined 4. The density of ρ_R at the point w_1 is given by

$$\lambda_R(w_1) = \inf\{\lambda_{\mathbb{D}}(z_1) \mid p(z_1) = w_1\}.$$

Definition

Let R be a hyperbolic Riemann surface. We define the first type Bloch space on R as

$$\mathcal{B}(R) := \left\{F \in \mathcal{O}(R) \mid \|F\|_{\mathcal{B}(R)} = \sup_{w \in R} \frac{|F'(w)|}{\lambda_R(w)}\right\} < \infty.$$

Definition

Let R be a hyperbolic Riemann surface with Green's function $g_R(z, z_0)$, by using local coordinates in a neighborhood of z_0, we can define the *Robin's constant* by

$$\gamma_R(z_0) = \lim_{z \to z_0} \left(g_R(z, z_0) - \log \frac{1}{|z - z_0|} \right).$$

Let $c_R(z_0) = e^{-\gamma_R(z_0)}$ be the *capacity density* of R at z_0.

Definition

Let R be a hyperbolic Riemann surface. We define the second type Bloch space on R as

$$\mathcal{CB}(R) := \left\{ F \in \mathcal{O}(R) \mid \|F\|_{\mathcal{CB}(R)} = \sup_{w \in R} \frac{|F'(w)|}{c_R(w)} < \infty \right\}.$$

Theorem

Let R be a hyperbolic Riemann surface, $Fuc(\mathbb{D})$ a Fuschian group such that $\mathbb{D}/Fuc(\mathbb{D})$ is biholomophic to R, and Ω the fundamental domain of $Fuc(\mathbb{D})$. Then

(i) $\mathcal{CB}(R) \subset \mathcal{B}(R)$ *i.e., there is a hyperbolic Riemann surface S such that*

$$\mathcal{CB}(S) \neq \mathcal{B}(S).$$

(ii) *If*

$$\delta(R) := \inf_{w\in\Omega}\left\{\prod_{\gamma\in Fuc(\mathbb{D})} |\sigma_w(\gamma(w))|\right\} > 0$$

then

$$\mathcal{CB}(S) = \mathcal{B}(S).$$

WHAT ELSE CAN I DO?

THANK YOU!

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