Colored Tilings on Graphs

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Original Problem (MSE)

Number of ways to partition $2 \times N$. Tile into m parts

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Given a $2 \times N$ Tile , how to find the number of ways to partition it into M parts ?

Meaning of a part : Cells having same number which are adjacent to each other form a part .

Lets take $N = 5$, so an example tile looks like :

1 1 1 2 2

3 3 2 2 1

This tile has 4-parts to it :

Part-1 : (1 1 1)

Part-2 :

```
(X 2 2)
(2 2 X)
```

Part-3 : (3 3)

Part-4 : (1)

Each cell can be assigned any number between 1 and K , how many ways exist to partition the tile into "M" parts ?

If we take each square of an $n \times m$ grid and associate one of k colors with probability $1/k$, what is the expected number of "Tetris pieces" we will see?

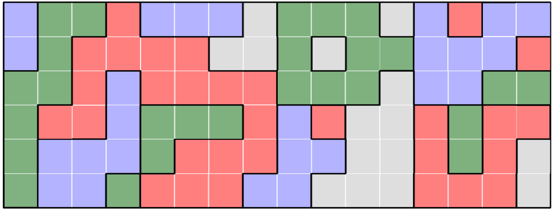


Figure: A 4-colored tiling element of $\mathcal{T}^{(4)}_{6,16}$.

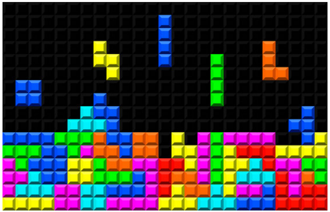
An old friend

Figure: José L. Ramírez (UNAL)

Object: Polyominoes

Def. Polyomino

In $\mathbb{Z} \times \mathbb{Z}$, a *cell* is a unitary square with integer coordinates. A *polyomino* is a finite collection of cells with connected interior joined edge to edge.



Tool: Generating Functions

Imagine you have a sequence of numbers i.e., F_n (The Fibonacci Numbers). You can generate them one by one using $F_n = F_{n-1} + F_{n-2}$ and $F_0 = 0$, $F_1 = 1$. What if there is a way to have them all at the same time just available for you?

Generating Function

It is a symbolic sum
$F(x) = F_0 + F_1 x + F_2 x^2 + \cdots + F_n x^n + \cdots$

$$F(x) = \frac{x}{1 - x - x^2}.$$

Operations Mean!

$$F(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \varphi x} - \frac{1}{1 - \bar{\varphi} x} \right)$$

$$F_n = \frac{\varphi^n - \bar{\varphi}^n}{\sqrt{5}}$$

Tool: Binomial Numbers

Binomial Numbers $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

$$(x+1)^n = x^n + nx^{n-1} + \binom{n}{2}x^{n-2} \cdots + \binom{n}{n}x^{n-n}$$

Eg:

$$(x+1)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2 + \binom{3}{2}x + \binom{3}{3} = x^3 + 3x^2 + 3x + 1.$$

They count ways to choose k elements out of n.

In how many ways can we express 5 as a sum of 3 integers? This is called **Composition**

$5 = 1 + 1 + 1 + 1 + 1$

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$$\binom{5-1}{3-1} = \binom{4}{2} = 6.$$

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Notation

$\mathcal{T}_{m,n}^{(k)}$ denotes the set of possible k-colored tilings in the $m \times n$ grid such that adjacent polyominos have different colors.
If T is k-colored in $\mathcal{T}_{m,n}^{(k)}$, $\rho(T)$ is the number of polyominos.

$$C_m^{(k)}(x,y) := \sum_{n \geq 1} x^n \sum_{T \in \mathcal{T}_{m,n}^{(k)}} y^{\rho(T)}. \qquad (1)$$

$c_{m,k}(n,i)$ will be the $x^n y^i$ coefficient of $C_m^{(k)}(x,y)$.

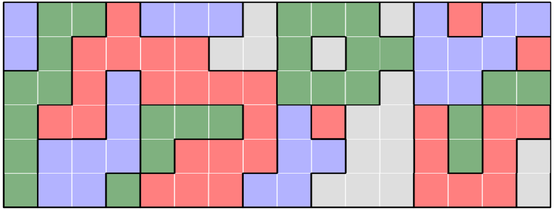

Figure: A 4-colored tiling element of $\mathcal{T}_{6,16}^{(4)}$.

Average number of polyominoes

Let $X_{\mathsf{Til}_{m,k}}(n)$ be a **random variable** that counts the number of polyominoes in a random tiling k-colored in $\mathcal{T}_{m,n}^{(k)}$. The following happens

$$\mathbb{E}[X_{\mathsf{Til}_{m,k}}(n)] = \frac{1}{\left|\mathcal{T}_{m,n}^{(k)}\right|} \sum_{T \in \mathcal{T}_{m,n}^{(k)}} \rho(T) = \frac{[x^n] \left. \frac{\partial C_m^{(k)}(x,y)}{\partial y} \right|_{y=1}}{[x^n] C_m^{(k)}(x,1)}, \tag{2}$$

where $[x^n]f(x)$ is the coefficient of x^n in $f(x)$.

Who cares?

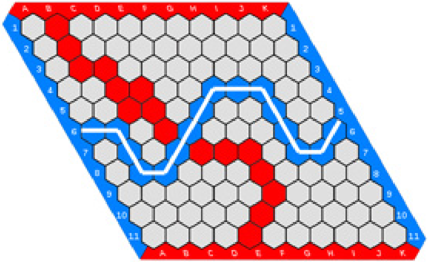

Figure: Hex game

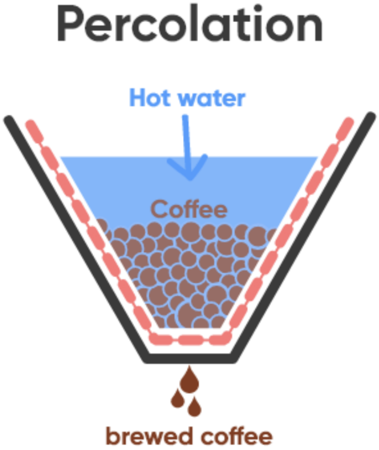

Figure: Percolation method

Hugo Cares

Figure: Hugo Duminil Copin. Fields Medal 2022

Case $m = 1$

These are **number compositions**! $c_{1,k}(n,i) = \binom{n-1}{i-1} k(k-1)^{i-1}$, then $C_1^{(k)}(x,y) = \frac{kxy}{1-x+xy-kxy}$.

Theorem

The exp. value of polyomonoes in $\mathcal{T}_{1,n}^{(k)}$ is

$$\mathbb{E}[X_{\mathrm{Til}_{1,k}}(n)] = \frac{(k-1)n+1}{k}.$$

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Case $m = 2$ (Original Problem)

Consider the following cases:

Let $\mathcal{A}_{2,n}^{(k)}$ and $\mathcal{B}_{2,n}^{(k)}$ the colored tilings in $\mathcal{T}_{2,n}^{(k)}$ s.t last colum has one or two colors. Then

$$A_2^{(k)}(x,y) := \sum_{n\geq 1} x^n \sum_{T\in\mathcal{A}_{2,n}^{(k)}} y^{\rho(T)} \quad \text{and} \quad B_2^{(k)}(x,y) := \sum_{n\geq 1} x^n \sum_{T\in\mathcal{B}_{2,n}^{(k)}} y^{\rho(T)}.$$

This means that $C_2^{(k)}(x,y) = A_2^{(k)}(x,y) + B_2^{(k)}(x,y)$.

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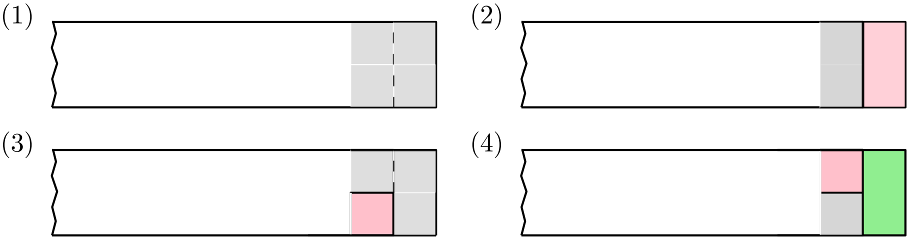


Figure: Decomposition of the colored tilings in $\mathcal{A}_{2,n}^{(k)}$.

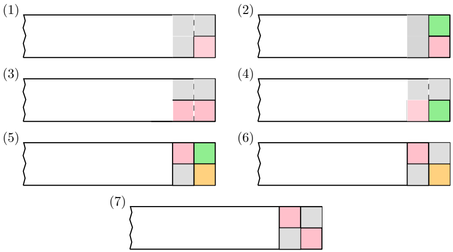
From this decomposition we get that

$$A_2^{(k)}(x,y) = kxy + \underbrace{xA_2^{(k)}(x,y)}_{(1)} + \underbrace{(k-1)xyA_2^{(k)}(x,y)}_{(2)} + \underbrace{2xB_2^{(k)}(x,y)}_{(3)} + \underbrace{(k-2)xyB_2^{(k)}(x,y)}_{(4)},$$

Case $m = 2$ (Original Problem)

$$B_2^{(k)}(x,y) = k(k-1)xy^2 + \underbrace{2(k-1)xyA_2^{(k)}(x,y)}_{(1)} + \underbrace{2\binom{k-1}{2}xy^2A_2^{(k)}(x,y)}_{(2)} + \underbrace{xB_2^{(k)}(x,y)}_{(3)}$$

$$+\underbrace{2(k-2)xyB_2^{(k)}(x,y)}_{(4)} + \underbrace{2\binom{k-2}{2}xy^2B_2^{(k)}(x,y)}_{(5)} + \underbrace{2\binom{k-2}{1}xy^2B_2^{(k)}(x,y)}_{(6)} + \underbrace{xy^2B_2^{(k)}(x,y)}_{(7)}.$$



Case $m = 2$ (Original Problem)

Theorem

The bivariate generating function $C_2^{(k)}(x, y)$ *is given by*

$$\frac{kxy(1 + (k-1)y - x(1-y)(1-ky))}{1 - x\left(2 + (3k-5)y + \left(k^2 - 3k + 3\right)y^2\right) + x^2(1-y)\left(1 - (k^2+1)y^2 - ky\left(1 - 2y\right)\right)}.$$

Furthermore, $[x^n]C_2^{(k)}(x, 1) = k^{2n}$*.*

Corollary

$$\mathbb{E}[X_{\mathrm{Til}_{2,k}}(n)] = \frac{2k^3n + k^2(2-3n) + n - 1}{k^3}.$$

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Case $m = 3$

Fundamental change

Now polyominoes can have **holes**!

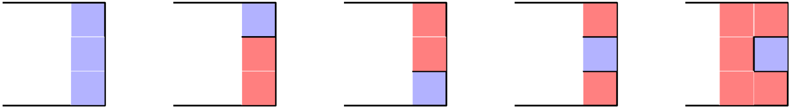

Figure: The 5 possible last columns of a configuration of size $3 \times n$.

Case $m = 3$

Theorem

The bivariate generating function $C_3^{(2)}(x,y)$ *is given by the quotient* $p(x,y)/q(x,y)$, *where*

$$p(x,y) = 2xy\left(\left(y^5 - 3y^4 + 7y^2 - 7y + 2\right)x^3 - \left(4y^5 - 6y^4 - y^3 + 10y^2 - 12y + 5\right)x^2 + \left(3y^4 - y^3 + y^2 - 3y + 4\right)x - (y+1)^2\right) \quad \text{and}$$

$$q(x,y) = (2y^5 - 7y^4 + 5y^3 + 5y^2 - 7y + 2)x^4 - (y^6 + 3y^5 - 7y^4 + 4y^3 + 5y^2 - 13y + 7)x^3 + (y^5 + 2y^4 + 3y^3 - y^2 - 6y + 9)x^2 - (y^3 + y^2 + 2y + 5)x + 1.$$

Corollary

$$\mathbb{E}\left[X_{\mathrm{Til}_{3,2}}(n)\right] = \frac{1183n + 1945 + 1/8^{n-2}}{1568}.$$

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Case $m = 3$ (number of fillings)

Prop.

The expected value for the number of fillings is

$$\frac{(7n - 15) + \frac{1}{8^{n-2}}}{1568}.$$

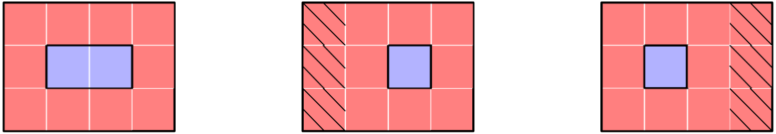

Figure: All possible configurations for tilings in $\mathcal{T}^{(2)}_{3,4}$ with exactly one filling.

General case

Question

Can we keep doing this?

Better question

How much do we have to suffer to get a system for $m = 4, 5, \cdots$? Notice that the size of the linear system is given by the **possible last columns**.

Even better question

How big of a computer do we need? or can I count the number of possible columns? How do we do this?

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Number of last columns (k = 2)

Idea

Let's do it for just two colors. Consider the column as a vector of $0'$s and $1'$s.

$$(c_1, c_2, c_3, \cdots, c_m) \in \{0, 1\}^m,$$

If $c_i = c_{i+1}$, then they belong to the same polyomino! Consider

$$(d_1, d_2, d_3, \cdots, d_\ell) \in \{0, 1\}^\ell,$$

s.t $\ell \leq m$, and $d_i \neq d_{i+i}$. Two options

$$(0, 1, 0, 1, \cdots, 0, 1),$$

$$(1, 0, 1, 0, \cdots, 1, 0).$$

Considering the other dimension, these points may be in the same polyomino!

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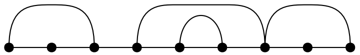
Other old friends

Idea

These points are joint by a wire if they belong to the same polyomino.



$$\{\{1,3\},\ \{2\},\ \{4,7,9\},\ \{5,6\},\ \{8\}\}$$

Figure: The wire diagram of a set partition.

Bell numbers

The number of set partitions of a set with n elements is given by the n-th Bell number

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k.$$

$B_n = 1, 1, 2, 5, 15, 52, 203, 877, 4140, \cdots$

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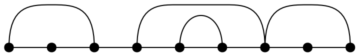

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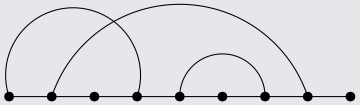
They are **not** all partitions

The wires cant cross!

Notice that we cant have the following scenario



Non-crossing partitions

The number of non-crossing partitions is given by the **n-th Catalan number**

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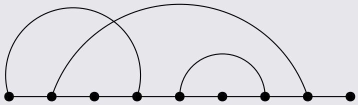

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Parity Condition

Notice we cant pair d_i with d_j if $i \not\equiv j \pmod 2$, because they have different colors!

Lemma

Let $NC(n)_0$ be the number of non-crossing partitions on n elements and that have the parity condition, then

$$|NC(n)_0| = \frac{1}{2\lfloor n/2 \rfloor + 1} \binom{n + \lfloor n/2 \rfloor}{\lceil n/2 \rceil}.$$

Fact of life

If we drop the non-crossing partition the number of such partitions is $B_{\lfloor n/2 \rfloor} \cdot B_{\lceil n/2 \rceil}$.

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Number of columns with two colors

Theorem

The number of different columns

$$\mathrm{Col}_{m,2} = 2\sum_{\ell=1}^{m} \frac{1}{2\lfloor \ell/2 \rfloor + 1} \binom{m-1}{\ell-1} \binom{\ell + \lfloor \ell/2 \rfloor}{\lceil \ell/2 \rceil}$$

$$= 0, \quad 2, \quad 4, \quad 10, \quad 26, \quad 72, \quad 206, \quad 608, \quad 1834, \quad 5636, \ldots$$

Corollary

$$\mathrm{Col}_{m,2} \sim c \cdot \frac{\left(1 + \frac{3}{2}\sqrt{3}\right)^m}{m^{3/2}},$$

where $c \approx 1.75213$.

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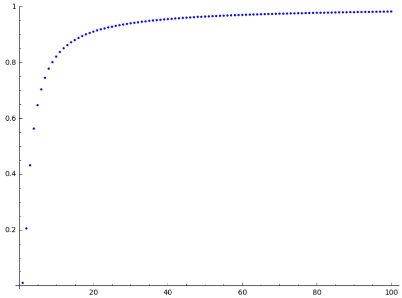
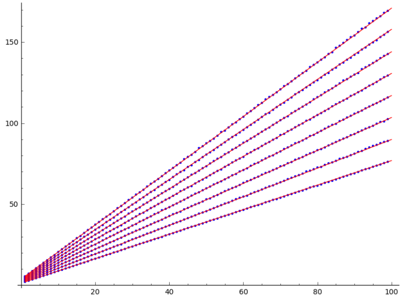
Experimentation

$k \backslash n$	3	4	5	6	7	8	9	10
2	0.756	0.889	1.015	1.147	1.280	1.410	1.544	1.671
3	1.409	1.778	2.150	2.521	2.889	3.257	3.629	4.000
4	1.779	2.299	2.815	3.329	3.836	4.357	4.874	5.382

Table: This contains values $m_{n,k}$ s.t $\mathbb{E}\left[X_{\mathrm{Til}_{n,k}}(x)\right] \sim m_{n,k} \cdot (x-1) + \frac{(k-1)n+1}{k}$.





So.. where are the graphs?

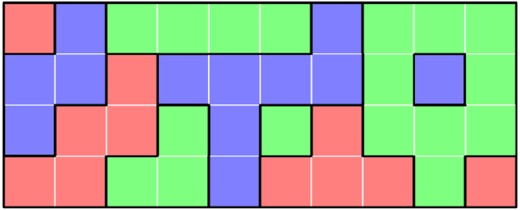


Figure: A tiling of the grid

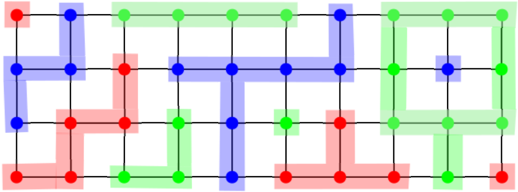


Figure: A partition of its graph

Notice that the grid is $P_m \times P_n$.

Take any family of graphs that you like

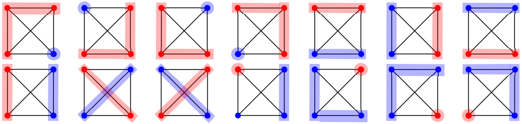
Theorem

The number of k-colored partitions of size i for the complete graph K_n, for $n \geq 1$, is given by

$$g_k(n,i) = \left\{ {n \atop i} \right\} \binom{k}{i} i!.$$

Corollary

$$\mathbb{E}\left[X_{\mathrm{Til}_k}(K_n)\right] = k - \frac{(k-1)^n}{k^{n-1}}.$$



Consider $G_m \times P_n$

Approach:

1. Create a bivariate generating function.

$$T_m^{(k)}(x, y) = \sum_{n \geq 1} x^n \sum_{T \in \mathcal{T}^{(k)}(U_n^{(m)})} y^{\rho(T)}.$$

2. Slice them and create a system of equations on them.
3. What's the size of the system?
4. Do some coding!
5. If the system is too big, use the symmetry of the graph and the colors!
6. Experiment and hope for the best.

Consider $G_m \times P_n$

Approach:

1. Create a bivariate generating function. $T_m^{(k)}(x, y)$
2. Slice them and create a system of equations on them.

$$\mathcal{C}_{m,k} = \{\mathcal{A} = (A_1, \ldots, A_k) : \cup_{i=1}^{k} A_i = [m] \text{ and } A_i \cap A_i = \emptyset\}.$$

$$T_{\mathcal{A}}(x, y) = xy^{|\mathrm{supp}(\mathcal{A})|} + x \sum_{\mathcal{B} \in \mathcal{C}_{m,k}} y^{|\{i \in [k] : A_i \neq \emptyset \text{ and } A_i \cap B_i = \emptyset\}|} T_{\mathcal{B}}(x, y).$$

3. What's the size of the system?
4. Do some coding!
5. If the system is too big, use the symmetry of the graph and the colors!
6. Experiment and hope for the best.

Consider $G_m \times P_n$

Approach:

1. Create a bivariate generating function. $T_m^{(k)}(x, y)$
2. Slice them and create a system of equations on them.

$$\mathcal{C}_{m,k} = \{\mathcal{A} = (A_1, \ldots, A_k) : \cup_{i=1}^{k} A_i = [m] \text{ and } A_i \cap A_i = \emptyset\}.$$

3. What's the size of the system? $|\mathcal{C}_{m,k}| = k^m$
4. Do some coding!
5. If the system is too big, use the symmetry of the graph and the colors!
6. Experiment and hope for the best.

Consider $G_m \times P_n$

Approach:

1. Create a bivariate generating function. $T_m^{(k)}(x, y)$
2. Slice them and create a system of equations on them.

$$\mathcal{C}_{m,k} = \{\mathcal{A} = (A_1, \ldots, A_k) : \cup_{i=1}^{k} A_i = [m] \text{ and } A_i \cap A_i = \emptyset\}.$$

3. What's the size of the system? $|\mathcal{C}_{m,k}| = k^m$
4. Do some coding! **Alert:** Solving systems is $O(\ell^3)$ for numbers.
5. If the system is too big, use the symmetry of the graph and the colors!

$$|\mathcal{C}_{m,k}/\sim| = [q^m]\binom{m+k}{k}_q$$

6. Experiment and hope for the best.

Consider $G_m \times P_n$

Approach:

1. Create a bivariate generating function. $T_m^{(k)}(x, y)$
2. Slice them and create a system of equations on them.

$$\mathcal{C}_{m,k} = \{\mathcal{A} = (A_1, \dots, A_k) : \cup_{i=1}^{k} A_i = [m] \text{ and } A_i \cap A_i = \emptyset\}.$$

3. What's the size of the system? $|\mathcal{C}_{m,k}| = k^m$
4. Do some coding!
5. If the system is too big, use the symmetry of the graph and the colors!

$$|\mathcal{C}_{m,k}/\sim| = [q^m]\binom{m+k}{k}_q$$

6. Experiment and hope for the best.

Consider $G_\ell \times P_n$

Theorem

The expected size of a k-colored partition on the graph $K_\ell \times P_n$ when you color uniformly and independently each vertex is given by

$$\mathbb{E}[X_{\mathrm{Til}_k}(K_\ell \times P_n)] = \frac{k^{\ell n-(2\ell-1)}\left(\left(k^{2\ell}-(k^2-1)^\ell\right)+(k-1)^\ell\left((k+1)^\ell-k^\ell\right)n\right)}{k^{\ell n}}.$$

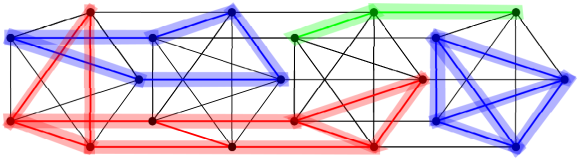


Figure: A 3-colored partition of size 4 of $K_5 \times P_4$.

Thanks!

References:

1. *Colored random tilings on grids*, J. L. Ramírez and D.V. J. Autom. Lang. Comb. (2024).
2. *Counting Colored Tilings on Grids and Graphs*, J.L Ramirez and D.V. Proceedings of GASCom 2024.
3. *Colored Tilings and Partitions on Graphs*, J. L. Ramirez and D.V. arXiv:2501.06008

Figure: SAGE experiments.

Questions

1. What is your favorite family of graphs? We have considered
   1.1 Trees
   1.2 Cycle graphs
   1.3 Complete Bipartite graphs
   1.4 Tadpole graphs (Undergraduate thesis Santiago Garcia, exp 2025).
2. Can we limit the size of the polyominos?
3. What is the prob. that a tile goes from first to last layer?