Dyck paths and nearly toric Schubert varieties

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JMM, AMS Combinatorics IV
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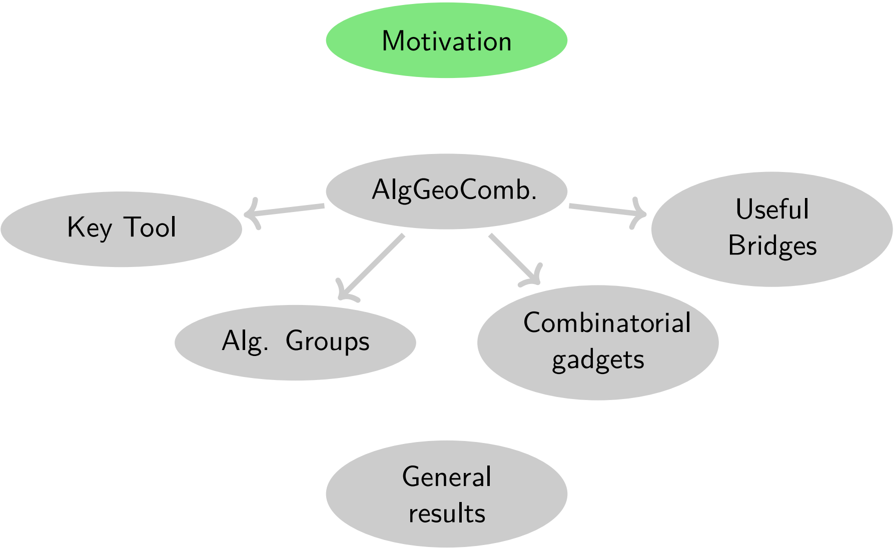
Outline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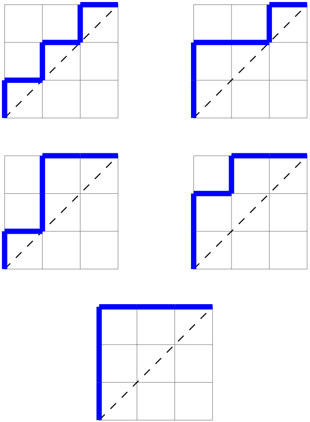
General results

Sacred number 😇



$\mathscr{L}^{+}_{3,3}$

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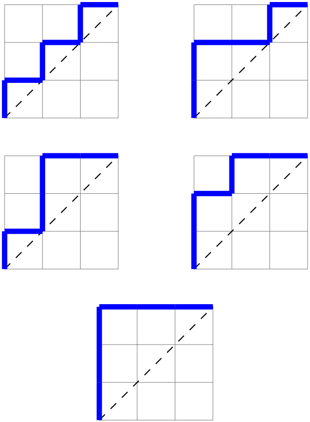

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$$\mathfrak{S}_3 := \left\{ w \mid w : [3] \xrightarrow{1:1} [3] \right\}$$
$$= \begin{Bmatrix} 123 & 213 & 132 \\ 231 & 312 & 321 \end{Bmatrix}.$$

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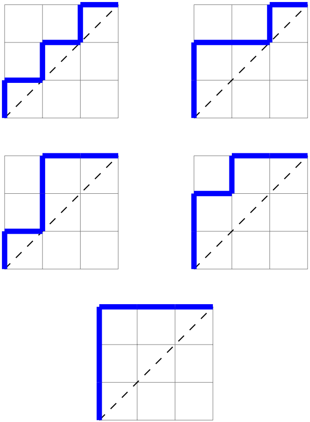

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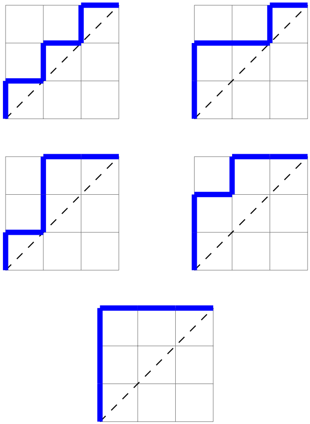

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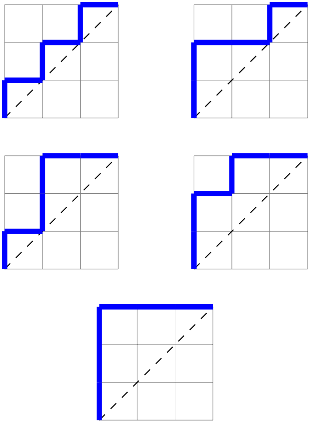

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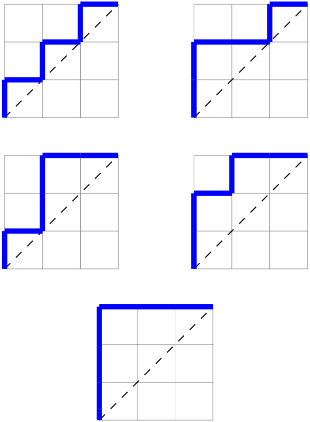

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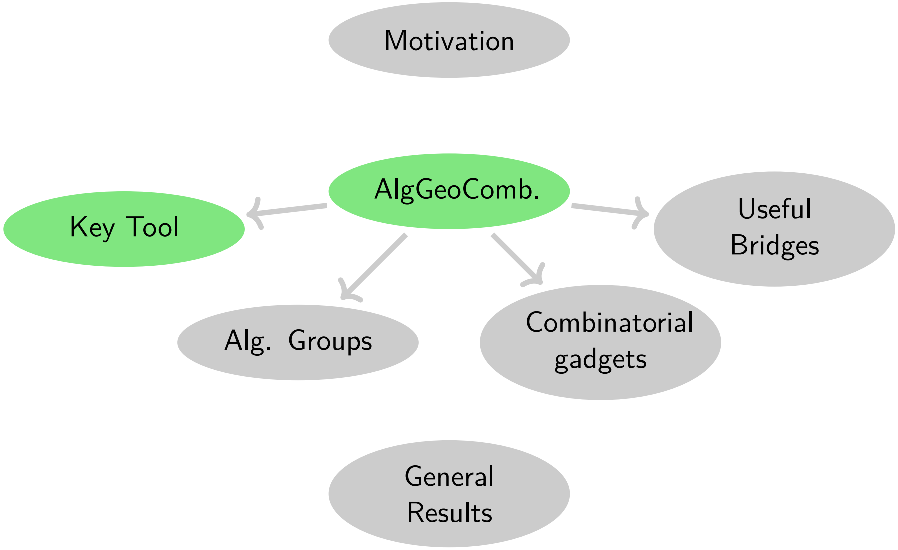
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Bandlow-Killpatrick (2001)

$(\mathfrak{S}_n, S)$ is a Coxeter system where $S = \{s_1, ..., s_{n-1}\}$ and $s_i = (i \quad i+1)$.

- Every w in $\mathfrak{S}_n$ can be written as product of the s_i's.
- If $w = s_{i_1} \cdots s_{i_\ell}$ and ℓ is minimal among all such expressions, then $\ell := \ell(w)$ is said to be the length of w, and the expression $s_{i_1} \cdots s_{i_\ell}$ is called a reduced decomposition for w.
  - $\mathrm{Red}(321) = \{s_1 s_2 s_1, s_2 s_1 s_2\}$

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Let $w = w_1 w_2 \cdots w_n$ be in $\mathfrak{S}_n^{312}$.

(1) Let x denote n. Find the index $i_1 \in [n]$ such that $w_{i_1} = x$. If $i_1 = x$, then move to the next step. If $i_1 \neq x$, then move the next step after computing the one-line expression for $ws_{i_1} s_{i_1+1} \cdots s_{n-1}$.

(2) Redefine w and x by setting $w := ws_{i_1} s_{i_1+1} \cdots s_{n-1}$ and $x := n-1$.

(3) If w is the identity, then stop the process. Hence, the expression $s_{n-1} s_{n-2} \cdots s_{i_1}$ is a reduced word for w
  - Otherwise, start over.

Example 😎

Let $w = w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 = 21654\mathbf{8}73$ be in $\mathfrak{S}_8^{312}$.

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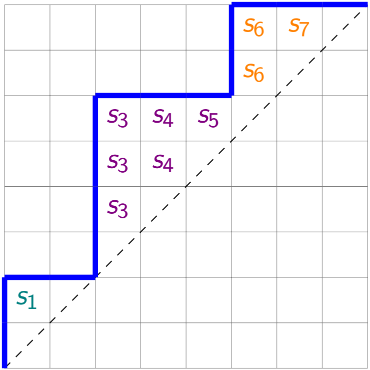

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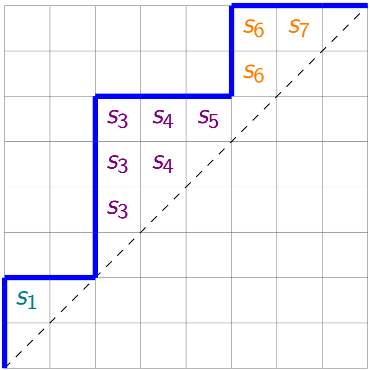

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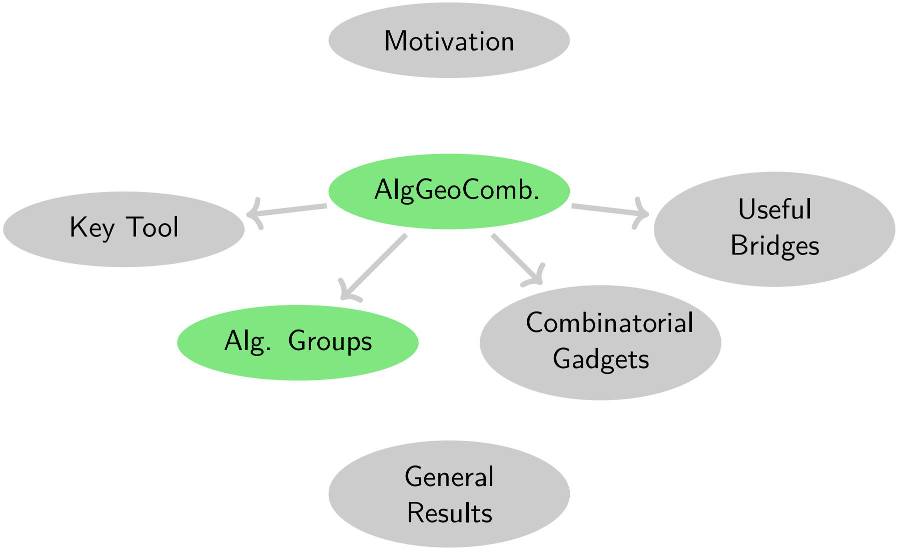
$$\mathfrak{S}_n^{312} \underset{\phi}{\overset{\psi}{\rightleftarrows}} \mathscr{L}_{n,n}^{+} \;;\; \ell(w) \longmapsto \mathrm{area}(\psi(w)) := \pi$$

Outline



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A little bit of abstractness

- Let **G** be a complex connected reductive algebraic group. A normal **G**-variety **Y** is called a spherical variety if it contains a dense orbit of some Borel subgroup $\mathbf{B} \subseteq \mathbf{G}$.

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- Ubiquitous examples:
  - Toric varieties
  - Wonderful varieties
  - Symmetric spaces

A little more …

- Let $\mathbf{G} = \mathrm{GL}_n$, $\mathbf{B}$, and $\mathbf{T}$ be the general linear group over $\mathbb{C}$, the subgroup of upper triangular matrices, and the diagonal matrices in $\mathbf{B}$ respectively.

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If $c_{\mathbf{T}}(X_{w\,\mathbf{B}}) = 1$, under what conditions $X_{w\,\mathbf{B}}$ is spherical with respect to some reductive group action?

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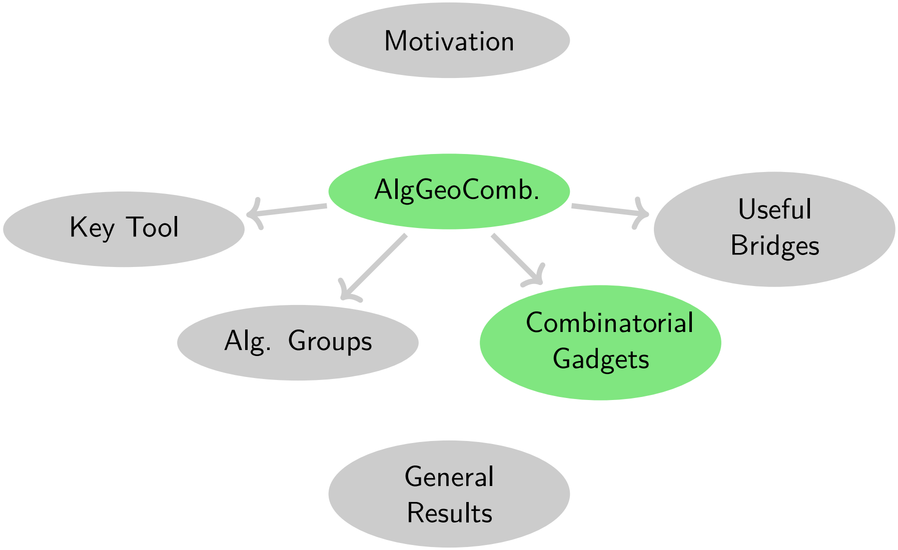

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Hence, we have a Borel subgroup $\mathbf{B}_{\mathbf{L}} \supset \mathbf{T}$ of $\mathbf{L}$ acting on $X_{w\,\mathbf{B}}$. We head towards *under what conditions* $c_{\mathbf{B}_{\mathbf{L}}}(X_{w\,\mathbf{B}}) = 0$*?*

Bruhat Order and Pattern Avoidance 😎

- The Bruhat–Chevalley order $(\mathfrak{S}_n, \leq)$ is defined by

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- We call $X_{w\,\mathbf{B}}$ a partition Schubert variety (Ding's Schubert variety) if w is a 312-avoiding permutation. Let us denote $\mathfrak{S}_n^{312}$ this family.
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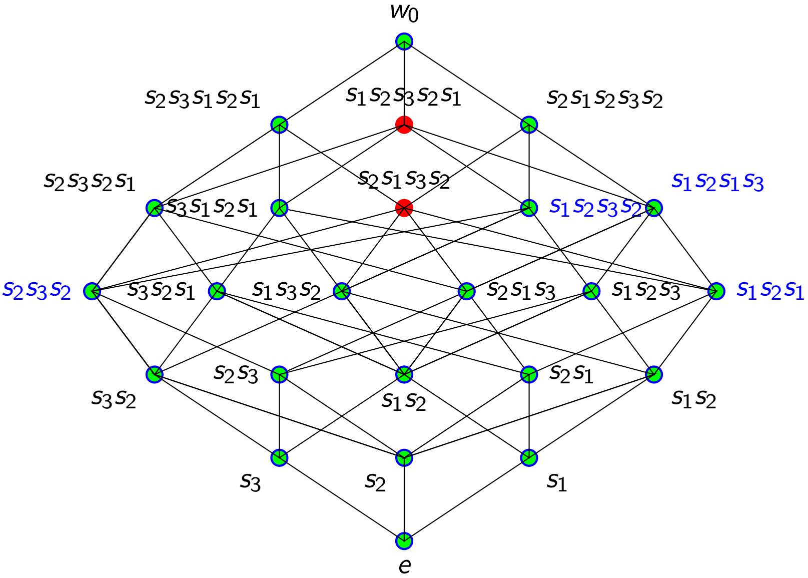
Theorem (Lakshmibai, Sandhya-1990)

The variety $X_{w\,\mathbf{B}}$ is smooth $\iff$ w avoids the patterns 3412 and 4231.



Bruhat order for $\mathfrak{S}_4$.

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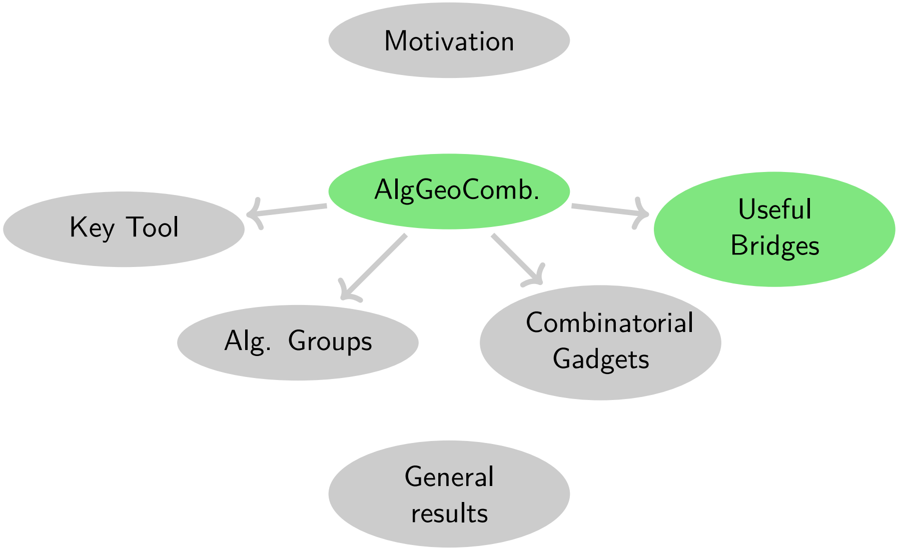

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Classification

Theorem (Lee, Masuda, Park-2021)

- $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ *and smooth* $\iff$ w *contains the pattern* 321 *exactly once and avoids* 3412 $\iff$ *there exists a reduced word of* w *containing* $s_i s_{i+1} s_i$ *as a factor and no other repetitions.*
- $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ *and singular* $\iff$ w *contains the pattern* 3412 *exactly once and avoids the pattern* 321.

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Theorem (Gaetz-2022)

$c_{\mathbf{B}_L}(X_{w\mathbf{B}}) = 0 \iff w$ *avoids the following 21 patterns*

$$\mathscr{P} := \begin{Bmatrix} 24531 & 25314 & 25341 & 34512 & 34521 & 35412 & 35421 \\ 42531 & 45123 & 45213 & 45231 & 45312 & 52314 & 52341 \\ 53124 & 53142 & 53412 & 53421 & 54123 & 54213 & 54231 \end{Bmatrix}$$

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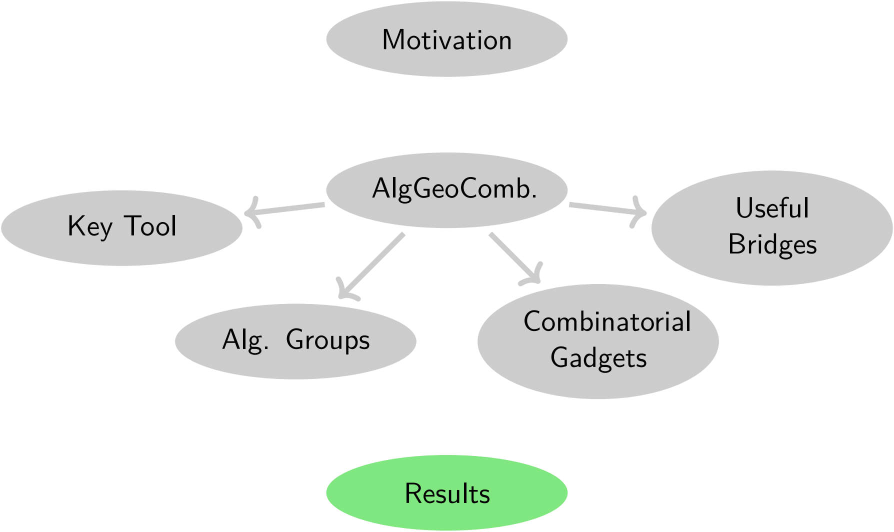
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First result ☕

Theorem (Can-D.)

(i) *Let* $\mathscr{D}_w \subset \mathrm{GL}_n / \mathbf{B}$ *be a partition Schubert variety such that* $c_{\mathbf{T}}(\mathscr{D}_w) = 1$. *Then* $\mathscr{D}_w$ *is a spherical* $\mathbf{L}$*-variety, where* $\mathbf{L}$ *is a Levi factor of the stabilizer of* $\mathscr{D}_w$ *in* GL_n.

- $\mathbf{NT}_n^{312} := \{w \in \mathfrak{S}_n^{312} : c_{\mathbf{T}}(X_{w\,\mathbf{B}}) = 1 \textit{ and } c_{\mathbf{B}_{\mathbf{L}}}(X_{w\,\mathbf{B}}) = 0\}$

First result ☕

Theorem (Can-D.)

(i) *Let $\mathscr{D}_w \subset \mathsf{GL}_n / \mathbf{B}$ be a partition Schubert variety such that $c_{\mathbf{T}}(\mathscr{D}_w) = 1$. Then $\mathscr{D}_w$ is a spherical $\mathbf{L}$-variety, where $\mathbf{L}$ is a Levi factor of the stabilizer of $\mathscr{D}_w$ in GL_n.*

- $\mathbf{NT}_n^{312} := \{w \in \mathfrak{S}_n^{312} : c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1 \textit{ and } c_{\mathbf{B}_{\mathbf{L}}}(X_{w\mathbf{B}}) = 0\}$

(ii) *All singular Schubert varieties of $\mathbf{T}$-complexity one are spherical.*

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Example

In the flag $\mathrm{GL}_5(\mathbb{C}) / \mathbf{B}_5(\mathbb{C})$, w is an element of

$$\mathbf{NT}_5^{312} = \begin{Bmatrix} 12543 & 13542 & 14325 & 14352 & 21543 & 23541 \\ 24315 & 24351 & 32145 & 32154 & 32415 & 32451 \end{Bmatrix}$$

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What's the cardinality of $\mathbf{NT}_n^{312}$?

Second result ☕

Theorem (Can-D.)

Let $\pi \in \mathscr{L}^{+}_{n,n}$ and $w = w_1 w_2 \cdots w_n$ be such that $\phi(\pi) = w$ for $n \geq 4$. Then w has a unique 321 if and only if π has a unique peak at the second diagonal and no other peaks at the r-th diagonal for $r \geq 3$.

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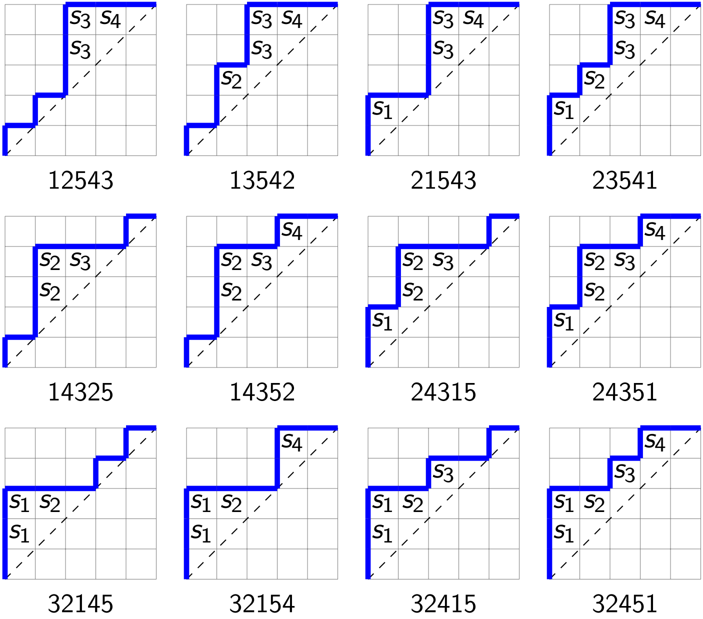
Corollary (Can-D.)

If $n \geq 4$, the cardinality of $\mathbf{NT}_n^{312}$ is given by $2^{n-3}(n-2)$.



Dyck paths of $\mathbf{NT}_5^{312}$.

Final result: new characterization 😋

Theorem (Can-D.)

Let w be in $\mathfrak{S}_n^{312}$. Let π denote the Dyck path of size n corresponding to w. Then $X_{w\mathbf{B}}$ is a spherical Schubert variety if and only if one of the following conditions holds:

Final result: new characterization 😛

Theorem (Can-D.)

Let w be in $\mathfrak{S}_n^{312}$. Let π denote the Dyck path of size n corresponding to w. Then $X_{w\mathbf{B}}$ is a spherical Schubert variety if and only if one of the following conditions holds:

(1) $\pi^{(2)} = \emptyset$, *or*

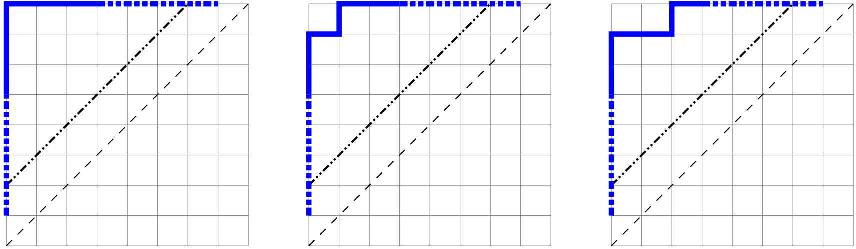
Final result: new characterization ☺

Theorem (Can-D.)

Let w be in $\mathfrak{S}_n^{312}$. Let π denote the Dyck path of size n corresponding to w. Then $X_{w\,\mathbf{B}}$ is a spherical Schubert variety if and only if one of the following conditions holds:

(1) $\pi^{(2)} = \emptyset$, *or*

(2) *every connected component M of $\pi^{(2)}$ is either an elbow, dimple, or ledge as depicted below*



Elbow Dimple Ledge

Thank You/Gracias/Obrigado 😄

https://arxiv.org/abs/2212.01234

"Stones on the road? I save every single one, and one day I'll build a castle." *Fernando Pessoa.*