From Differential Geometry to Lie Theory

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Outline

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- The French mathematician Henri Poincaré developed more fundamental tools in topology and homology. It helped to Hermann Weyl to give the current definition of manifold in 1913 on his *Die Idee Riemannschen Fläche*.

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Cartography shows us the interplay of two geometric “objects” which locally “preserves” information.

(a) The Globe $\subseteq \mathbb{R}^3$

(b) Colombia $\subseteq \mathbb{R}^2$

What I realized when I was a kid.

Definition

Let M be a topological space. We say that M is topological n-manifold if it has the following properties:

- M is a **Hausdorff space**: $\forall p, q \in M, \exists \mathcal{U}_p, \mathcal{V}_q \subseteq M \text{ s.t. } \mathcal{U}_p \cap \mathcal{V}_q = \emptyset$.

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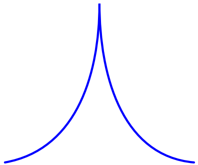
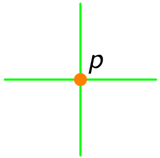
- M is **second-countable**:There exists a countable basis for the topology of M.
- M is **locally-Euclidean of dimension** n: For each $p \in M$, we can find
  - an open subset $U \subseteq M$ containing p,
  - an open subset $V \subseteq \mathbb{R}^n$,
  - a *homeomorphism* $\phi : U \to V$ which is said to be a chart.

Example





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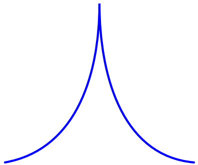

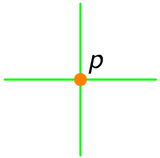

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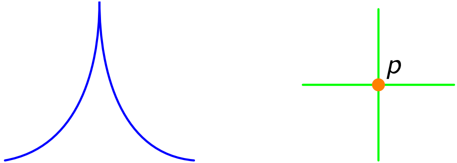

- The cusp is a topological manifold since it is subspace of $\mathbb{R}^2$.
- The cross is not topological manifold with the subspace topology at p since it is not locally Euclidean at p.

• Let (U, φ) and (V, ψ) two charts on M such that $U \cap V \neq \emptyset$. The composite map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is said to be the transition map.

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• A smooth atlas on a manifold M is a family $\mathfrak{A} = \{U_\alpha, \varphi_\alpha\}$ of charts such that

- are differentiable or C^∞ compatible on M,
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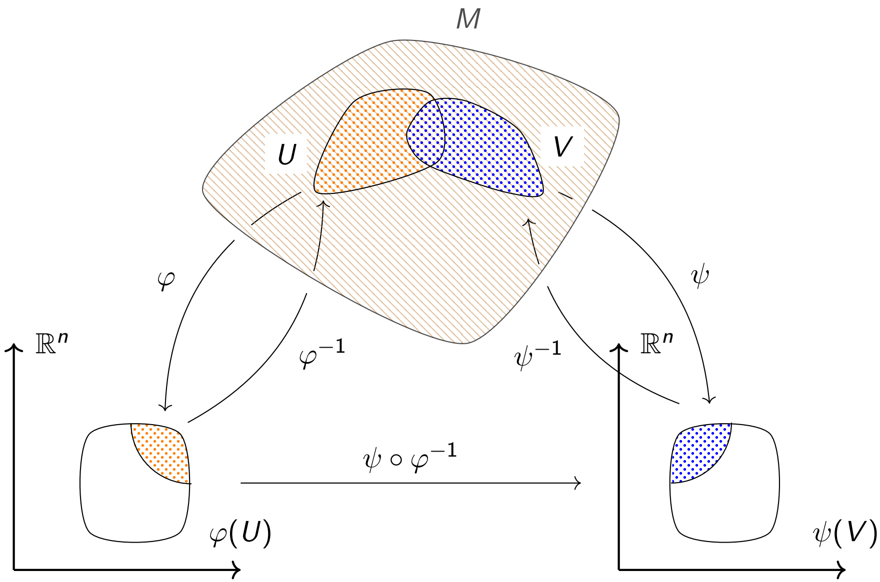
• A smooth manifold is a pair $(M, \mathfrak{A})$ where M manifold and $\mathfrak{A}$ is a smooth structure on M.


M
U
V
φ
φ^{-1}
ψ
ψ^{-1}
$\mathbb{R}^n$
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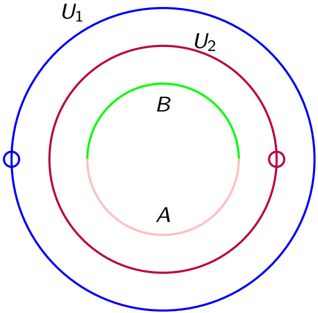
$\varphi(U)$
$\psi(V)$

Let us consider the unit circle

$$\mathbb{S}^1 := \{e^{it} \in \mathbb{C} \mid t \in [0, 2\pi]\},$$

and the following open subsets of $\mathbb{S}^1$:

$$\begin{cases} U_1 := \{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\}; & \varphi_1(e^{it}) = t \\ U_2 := \{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\}; & \varphi_2(e^{it}) = t \end{cases}.$$



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- The transitions map are given

$$(\varphi_2 \circ \varphi_1^{-1})(t) = \begin{cases} t + 2\pi & t \in (-\pi, 0) \\ t & t \in (0, \pi) \end{cases},$$

and

$$(\varphi_1 \circ \varphi_2^{-1})(t) = \begin{cases} t - 2\pi & t \in (\pi, 2\pi) \\ t & t \in (0, \pi) \end{cases}$$

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In practice, we don't need to exhibit the maximal atlas. The existence of *any* atlas on M will be sufficient:

Proposition

Any atlas $\mathfrak{A} = \{(U_\alpha, \psi_\alpha)\}$ *on a locally Euclidean space is contained in a unique maximal atlas.*

- Any open subset V of a M is a smooth manifold as well. Let $\{(U_\alpha, \psi_\alpha)\}$ be an atlas for M, then

$$\{(U_\alpha \cap V, \psi_\alpha|_{U_\alpha \cap V})\} \quad \text{where} \quad \psi_\alpha|_{U_\alpha \cap V} : U_\alpha \cap V \to \mathbb{R}^n$$

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- Since $\mathbf{k}^n \times \mathbf{k}^n \cong \mathbf{k}^{nm}$, the general linear group

$$\mathrm{GL}(n, \mathbf{k}) := \{A \in \mathbf{k}^{n \times n} : \det A \neq 0\}$$

is a smooth manifold. Indeed, $\det : \mathbf{k}^{n \times n} \to \mathbf{k}$ is continuous and so $\mathrm{GL}(n, \mathbf{k})$ is an open subset of $\mathbf{k}^{n^2}$ where $\mathbf{k}$ is either $\mathbb{R}$ or $\mathbb{C}$.

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- The product of two smooth manifolds is also a smooth manifold. For instance, $\mathbf{T} \times \mathbf{T} = \mathbf{T}^2$ is a smooth manifold.

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- Let M and N two smooth manifold and let $F : M \to N$ be any map. F is said to be a smooth map if for every $p \in M$, there exist smooth charts (U, φ) containing p and (V, ψ) containing $F(p)$ such that $F(U) \subseteq V$ and the composition $\psi \circ F \circ \varphi^{-1}$ is smooth from $\varphi(U)$ to $\psi(V)$.

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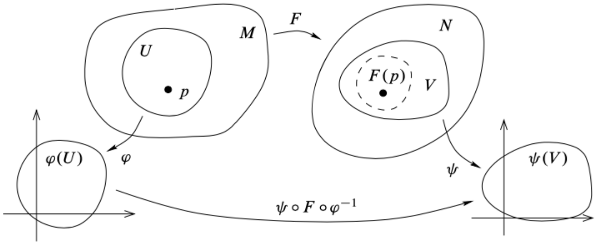
    - The map $\varepsilon : \mathbb{R}^n \to \mathbf{T}^n$ defined by $\varepsilon(x^1, ..., x^n) = (e^{2\pi i x^1}, ..., e^{2\pi i x^n})$.



- A diffeomorphism from M to N is a smooth bijective map $F : M \to N$ that has a smooth inverse.
  - Consider $F : \mathbb{B}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{B}^n$ by

$$F(x) = \frac{x}{\sqrt{1 - |x|^2}}, \qquad G(y) = \frac{y}{\sqrt{1 + |y|^2}}.$$

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Theorem (Dimension-Invariance)

A nonempty smooth manifold of dimension m cannot be diffeomorphic to an n-dimensional smooth manifold unless $m = n$.

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- Consider two smooth structures $(\mathbb{R}, \mathrm{id})$ and $(\mathbb{R}, \psi)$ where $\psi(x) = x^3$ on $\mathbb{R}$. These are not compatible since the transition map $\mathrm{id} \circ \psi^{-1}(y) = y^{1/3}$ is not smooth at the origin.

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- Define $F : \mathbb{R} \to \mathbb{R}$ by $F(x) = x^{1/3}$. The local coordinate of this map and its inverse are

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  which are smooth. Thus, these smooth structures are not diffeomorphic.
- There is only one smooth structure on $\mathbb{R}$ up to diffeomorphism!! ☕

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- Later, Milnor and Kerviare showed that there are 28 diffeomorphism classes of such structures.
- The problem of identifying the number of smooth structures (if any) on a topological 4-dimensional is open...

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- A linear map $X_p : C^\infty(M) \to \mathbb{R}$ is called a derivation at p if it satisfies

$$X_p(fg) = (X_p f)g(p) + f(p)(X_p g) \quad \text{for all} \quad f, g \in C^\infty(M).$$

The set of all derivation of $C^\infty(M)$ at p, denoted by T_pM, is a vector space called the tangent space to M at p.

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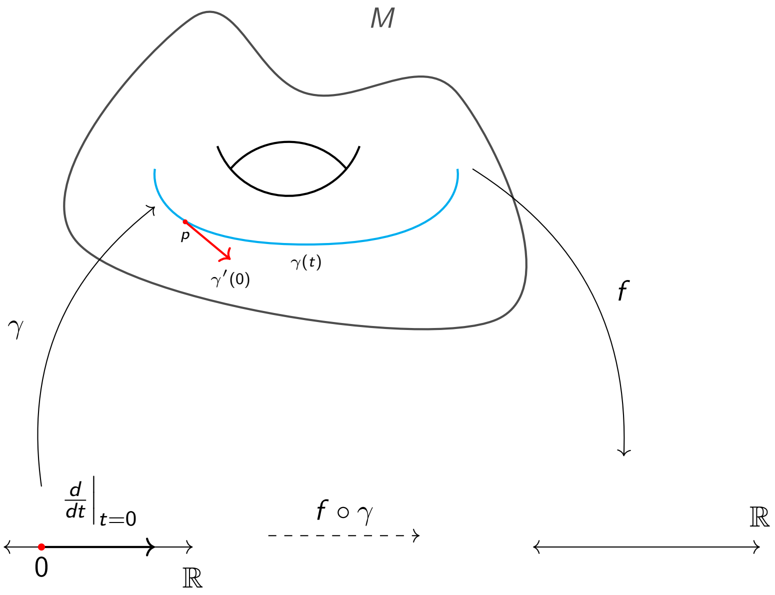
- The tangent space T_pM is the set of all linear maps $v : C^\infty(M) \to \mathbb{R}$ of the form

$$v(f) = \left.\frac{d}{dt}\right|_{t=0} (f \circ \gamma)(t),$$

  for some $\gamma \in C^\infty(I, M)$ with $\gamma(0) = p$.


M
p
$\gamma'(0)$
$\gamma(t)$
γ
f
$\frac{d}{dt}\Big|_{t=0}$
0
$\mathbb{R}$
$f \circ \gamma$
$\mathbb{R}$

- The differential or pushforward of F at p is said to be

$$
\begin{array}{ccc}
M & \xrightarrow{\quad F \quad} & N \\
\big\downarrow & & \big\downarrow \\
T_pM & \xrightarrow[\ dF_p\ ]{} & T_{F(p)}N
\end{array}
\qquad dF_p(X)(f) := X(f \circ F).
$$

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$$\begin{array}{ccc} M & \xrightarrow{F} & N \\ \downarrow & & \downarrow \\ T_pM & \xrightarrow[dF_p]{} & T_{F(p)}N \end{array} \qquad dF_p(X)(f) := X(f \circ F).$$

- Properties:
  - $dF_p : T_pM \to T_{F(p)}N$ is linear.
  - $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_pM \to T_{G \circ F(p)}P$.
  - $d(\mathrm{id}_m)_p = \mathrm{id}_{T_pM} : T_pM \to T_pM$.
  - F Diffeo, $dF_p : T_pM \to T_{F(p)}N$ is an ISO and $(dF_p)^{-1} = d(F^{-1})_{F(p)}$.
  - $\dim T_pM = \dim M$.

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In other words, the Jacobian matrix

$$dF_p = \begin{pmatrix} \frac{F^1}{\partial x^1}(p) & \cdots & \frac{\partial F^1}{\partial x^n(p)} \\ \vdots & \ddots & \vdots \\ \frac{F^m}{\partial x^1}(p) & \cdots & \frac{F^m}{\partial x^n}(p) \end{pmatrix}$$

It turns out our knowledge from linear algebra can be used:

- Pick $F \in C^\infty(M, N)$ and $p \in M$. The rank of F at p is defined as

  $\mathrm{rank}_p(F) = \mathrm{rank}_{\varphi(p)}(\psi \circ F \circ \varphi^{-1})$; $\mathrm{rank}(T) := \dim(\mathrm{span}\{T(\beta)\})$

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- Smooth immersion whether its differential is injective at each point i.e. $\mathrm{rank}\, F = \dim M$.
  - Smooth embedding whether F is an injective immersion which is a homeomorphism of M onto its image. The image of an embedding is called an imbedded submanifold

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- $\lambda : (-\pi, \pi) \to \mathbb{R}^2, \quad t \mapsto (\sin(2t), \sin t)$ is an immersion and one-to-one. However, it is not a homeomorphism into its image under the subspace topology...

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- Let $F \in C^\infty(M, N)$ be a smooth map. A point $q \in N$ is said to be a regular value of F if for all $x \in F^{-1}(q)$, one has $\mathrm{rank}_x(F) = \dim N$. It is a said to be singular value if it is not a regular value.

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• Let $F \in C^\infty(\mathbb{R}^{n+1}, \mathbb{R})$ be a smooth map defined as $F(x^0, ..., x^n) = (x^0)^2 + \cdots + (x^n)^2$. The Jacobian is $dF_p = (2x^0, ..., 2x^n)$. Thus, all the level set $F^{-1}(q)$ are submanifolds of $\dim S = (n+1) - 1 = n$.

- Let $\mathsf{O}(n,\mathbf{k}) := \{A \in \mathsf{GL}(n,\mathbf{k}) \mid AA^* = I_n\}$ be the orthogonal "group". Consider $F : \mathsf{Mat}(n,\mathbb{R}) \to \mathsf{Sym}(n,\mathbb{R}); \quad F(A) = AA^t, \quad \text{and} \quad F^{-1}(I_n) = \mathsf{O}(n,\mathbb{R})$

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$$\dim \mathrm{O}(n,\mathbb{R}) = \dim \mathrm{Mat}(n,\mathbb{R}) - \dim \mathrm{Sym}(n,\mathbb{R}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

A Lie group is a smooth manifold G that also is a group such that the two group operations

$$\mu : G \times G \to G; \quad \mu(a, b) = ab; \quad \text{and} \quad i : G \to G; \quad i(a) = a^{-1}$$

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- The $(\mathrm{GL}(n, \mathbf{k}), \cdot)$ is a Lie group.
- $(\mathbf{k}^{\times}, \cdot)$ is a Lie group.
- $\mathbb{S}^1 \subseteq \mathbb{C}^{\times}$ is a Lie group.
- Any group with discrete topology is a Lie group.

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Theorem (Closed Subgroup ☕)

Let G be a Lie group and $H \subseteq G$ a subgroup. Then H is a smooth embedding (regular) Lie subgroup $\iff$ H is closed.

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$$\det : \mathrm{GL}(n, \mathbf{k}) \to \mathbf{k}^{\times} \Rightarrow \ker(\det) = \mathrm{SL}(n, \mathbf{k}), \quad \dim \mathrm{SL}_n(\mathbf{k}) = n^2 - 1.$$

If G is a group and M a manifold. An action of G on M is a smooth map from $\theta : G \times M \to M$ such that

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- The action os said to be free if every isotropy group is trivial.

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Same method can be used to prove that $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ are connected Lie groups.

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A vector field on M is a section of the map $\pi : TM \to M$. More precisely, a vector field is a continuous map $X : M \to TM$, written $p \mapsto X_p$ such that

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$$\text{e.g} \qquad X_{(x,y)} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

defines a a *smooth* vector field on $\mathbb{R}^2$.

Given $X, Y \in \mathfrak{X}(M)$, the composition $X \circ Y$ is not a vector field:

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- For any two vectors fields $X, Y \in \mathfrak{X}(M)$ the commutator

$$[X, Y] := X \circ Y - Y \circ X : C^\infty(M) \to C^\infty(M)$$

is called the Lie Bracket.

Given $X, Y \in \mathfrak{X}(M)$, the composition $X \circ Y$ is not a vector field:

$$X = Y = \frac{\partial}{\partial x} \Longrightarrow X \circ Y = \frac{\partial^2}{\partial x^2} \quad \text{Leibniz multiplication} \rightarrow\leftarrow$$

- For any two vectors fields $X, Y \in \mathfrak{X}(M)$ the commutator

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Let k be a field. A Lie algebra over k is a a vector space $\mathfrak{g}$ over k together with a product $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the latter properties.

We parse

$$\begin{array}{ccc} G & \xrightarrow{\;L_g\;} & G \\ \big\downarrow\!\wr & & \big\downarrow\!\wr \\ T_eG & \xrightarrow[\;dL_g\;]{} & T_gG \end{array}$$

Thus, if we describe the tangent space T_eG at the identity, then dL_g will give us information at any point g.

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- Consider $X \in T_I\,\mathrm{SL}(n,\mathbb{R})$. There is a curve $\gamma : (-\epsilon,\epsilon) \to \mathrm{SL}(n,\mathbb{R})$ with $\gamma(0) = I$ and $\gamma'(0) = X$. In particular,

$$\det \gamma(t) = 1$$

for all $t \in (-\epsilon,\epsilon)$.

Thereby,

$$\begin{aligned}
\frac{d}{dt}\det(\gamma(t))\Big|_{t=0} &= d(\det\circ\gamma)\left(\frac{d}{dt}\Big|_{t=0}\right) \\
&= d(\underset{I}{\det})\left(d\left(\gamma\frac{d}{dt}\Big|_{t=0}\right)\right) \\
&= d(\underset{I}{\det})(\gamma'(0)) \\
\det(e^X) = e^{\operatorname{tr} X} \qquad &= \operatorname{tr}(X)
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$$

It turns out the Lie algebra of the Lie group $\mathrm{SL}(n\, R)$ is given by

$$
\mathfrak{sl}(n,\mathbb{R}) = \{X \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{tr}(X) = 0\}, \quad \dim\mathfrak{sl}(n,\mathbb{R}) = n^2 - 1.
$$

- Let G be a Lie group. A vector field $X \in \mathfrak{X}(G)$ is called left invariant whether $(dL_g)X = X$ for all $g \in G$. Let $L(G)$ be the vector space of all left-invariant vector fields on G. So,

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Namely, the isomorphism

$$T_eG \cong L(G)$$

allows is to define a Lie bracket on T_eG and to push forward left-invariant vector fields under a Lie homomorphism.

- For any real or complex Lie group G, there is a bijection between connected Lie subgroups $H \subseteq G$ and Lie subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$ given by $H \to \mathfrak{h} = T_e H \cong L(H) := Lie(H)$.

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- Any finte-dimensional real or complex Lie algebra is isomorphic to a Lie group.

$$\begin{array}{ccc} \mathfrak{g}_1 & \xrightarrow{d\phi} & \mathfrak{g}_2 \\ \Big\downarrow \exp & & \Big\downarrow \exp \\ G_1 & \xrightarrow[\phi]{} & G_2 \end{array}$$

Thank You/Gracias!

"We never love anyone. What we love is the idea we have of someone. It's our own concept-our own selves-that we love". *Fernando Pessoa*

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