Manifolds and its “applications”

Néstor Fernando Díaz Morera
Advisers
Dr. Lino Feliciano Reséndis Ocampo
Dr. Luis Manuel Tovar Sánchez

Instituto Politécnico Nacional, ESFM, Ciudad de México
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Outline

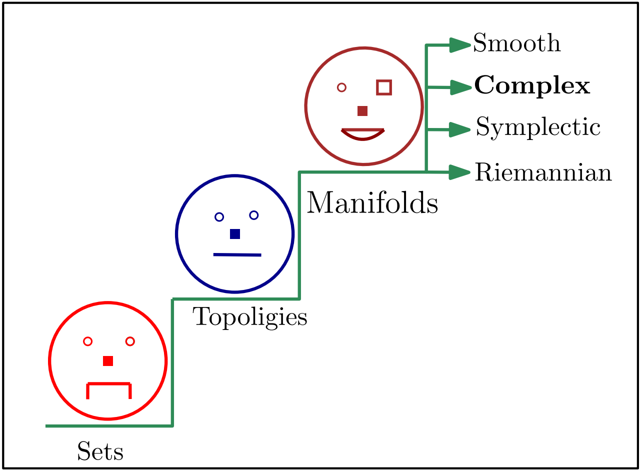

Figura: Timeline by Fernando.

Definition

Let M be a topological space, then

- M is **path-connected** if for any two points $a, b \in M$ there is a continuous function $f : [0, 1] \to M$ such that $f(0) = a$ and $f(1) = b$.
- M is **simply connected** if it is path-connected and any continuous map $g : \mathbb{S}^1 \to M$ can be contracted to a point.

Example

- Any **convex** set $B \subset \mathbb{R}^n$ is simply connected.
- $\mathbb{R}^n$ itself is simply connected.
- For $n \geq 2$, $\mathbb{S}^n$ is simply connected.

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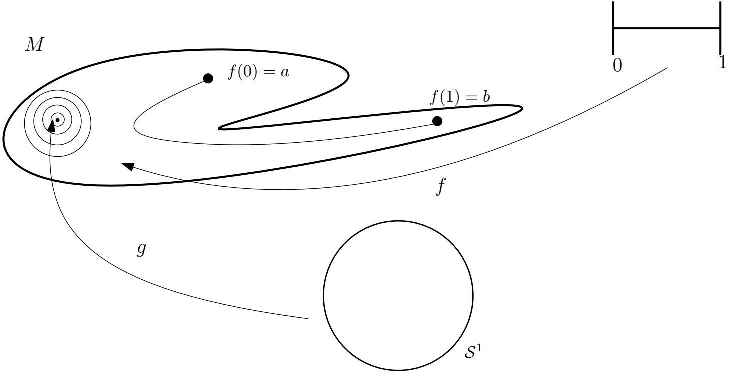

Figura: Path and simply connected spaces.

Definition

Let M be a topological space. We say that M is a **topological** $n-$**manifold** if it has the following properties:

- M is a **Hausdorff space**: for every pair of distinct points $p, q \in M$, there are disjoint open subsets $U, V \subseteq M$ such that $p \in U$ and $q \in V$.
- M is **second-countable**: there exists a countable basis for the topology of M.
- M is **locally Euclidean of dimension** n: each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.

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$\mathbb{R}^n$ is a $n-$manifold since

- It is Hausdorff because it is a metric space.
- It is second-countable because the set of all open balls with rational centers and rational radii is a countable basis for its topology.

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Let M be a two dimensional manifold. A **complex chart** on M is a homeomorphism $\varphi : U \to V$ of an open subset $U \subset M$ onto an open subset $V \subset \mathbb{C}$. The open subset U is called the *domain* of the chart φ. The chart φ is said to be *centered at* $p \in U$ if $\varphi(p) = 0$.

Definition

Two complex charts $\varphi_i : U_i \to V_i$, $i = 1, 2$ on M are said to be **compatible** if either $U_1 \cap U_2 = \emptyset$ or the map

$$\varphi_2 \circ \varphi_1^{-1} : \varphi(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2).$$

is *holomorphic* (See Figure).

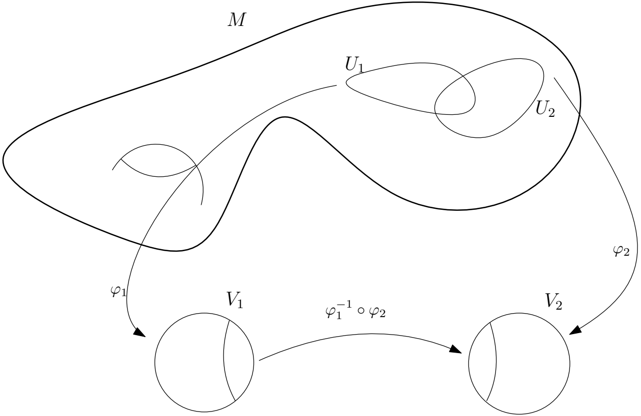

Figura: Compatible charts.

Definition

A **complex atlas** (or simply *atlas*) $\mathfrak{A}$ on M is a collection $\mathfrak{A} = \{\varphi_i : U_i \to V_i, i \in I\}$ of pairwise compatible complex charts whose domains cover M, i.e., $M = \bigcup_{i \in I} U_i$.

Definition

Two complex atlases $\mathfrak{A}$ and $\mathfrak{B}$ are *equivalent* if every chart of one is compatible with every chart of the other respectively.

Definition

A **complex structure** on M is a maximal complex atlas on M, or, equivalently, an equivalence class of complex atlases on M.

Definition (**The definition of a Riemann Surface**)

A *Riemann surface* is a pair (M, Σ), where M is a connected two-dimensional manifold and Σ is a complex structure on M.

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Definition (**The definition of a Riemann Surface**)

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Example

Let $M = \mathbb{C}$, and let U be any open subset. Define $\varphi_U(x, y) = x + iy$ from (considered as a subject of $\mathbb{C}$) to the complex plane. This is a complex chart on $\mathbb{C}$. Moreover Let M be $\mathbb{C}$ itself, considered topologically as $\mathbb{R}^2$. Therefore, it is a Riemann surface which is called *complex plane*.

Example (**Sphere.**)

Let $\mathbb{S}^2$ denote the unit 2-sphere inside $\mathbb{R}^3$, i.e.,

$$\mathbb{S}^2 = \{(x, y, t) \in \mathbb{R}^3 | x^2 + y^2 + t^2 = 1\}.$$

Consider the $t = 0$ plane as a copy of the complex plane $\mathbb{C}$, with $(x, y, 0)$ being identified with $z = x + iy$.

Example (carrying on...)

Let's us considere the following two charts

$$U_1 = \mathbb{S}^2 \setminus \{(0,0,1)\}, \quad \varphi_1(x,y,t) = \frac{x}{1-t} + i\frac{y}{1-t}$$
$$U_2 = \mathbb{S}^2 \setminus \{(0,0,-1)\}, \quad \varphi_2(x,y,t) = \frac{x}{1+t} - i\frac{y}{1+t}$$

Since $\dfrac{x-iy}{1+t} = \dfrac{1-t}{x+iy}$, it follows that the transition function is

$$\varphi_2 \circ \varphi_1^{-1}(z) = \frac{1}{z}$$

which is holomorphic on a domain $\varphi_1(U_1 \cap U_2) = \mathbb{C} \setminus \{0\}$.

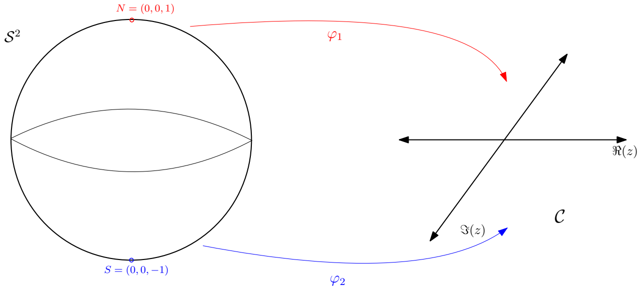


Figura: Compatible charts on $\mathbb{S}^2$.

Example (**Riemann Sphere**)

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the *one point compactification* of $\mathbb{C}$. Thus, $\widehat{\mathbb{C}}$ is compact Hausdorff topological such that $\widehat{\mathbb{C}} \simeq \mathbb{S}^2$. So, the complex structure is given by:

$$\begin{aligned} U_1 &= \mathbb{C}, & \varphi_1(z) &= z \\ U_2 &= \widehat{\mathbb{C}} \setminus \{0\}, & \varphi_2(z) &:= \begin{cases} 1/z \text{ if } z \neq \infty \\ 0 \text{ if } z = \infty. \end{cases} \end{aligned}$$

Since $\varphi_1(U_1 \cap U_2) = \varphi_2(U_1 \cap U_2) = \mathbb{C} \setminus \{0\} = \mathbb{C}^*$, it follows

$$\varphi_2 \circ \varphi_1^{-1} : \mathbb{C}^* \to \mathbb{C}^*, \quad z \mapsto \tfrac{1}{z}$$

is holomorphic.

Definition

Let M be a Riemann surface and $Y \subset M$ a open subset. A function $f : Y \to \mathbb{C}$ is called **holomorphic**, if for every chart $\psi : U \to V$ on M the function

$$f \circ \psi^{-1} : \psi(U \cap Y) \to \mathbb{C}$$

is holomorphic in the usual sense on the open set $\psi(U \cap Y) \subset \mathbb{C}$.

Definition

Suppose M and N are Riemann surfaces. A continuous map $F : M \to N$ is called *holomorphic*, if for every pair of charts $\psi_1 : U_1 \to V_1$ on M and $\psi_2 : U_2 \to V_2$ on N with $f(U_1) \subset U_2$, the mapping

$$\psi_2 \circ F \circ \psi_1^{-1} : V_1 \to V_2$$

is holomorphic in the usual sense.

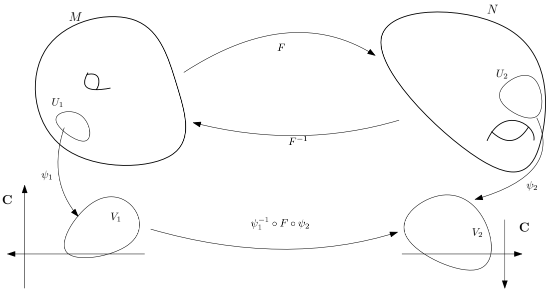
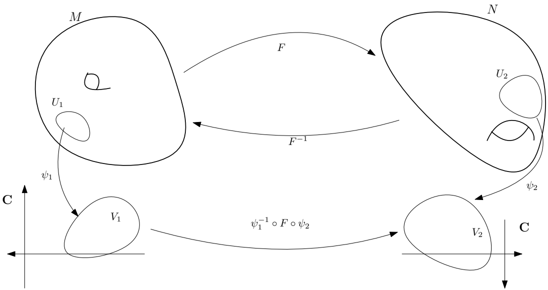


Figura: Morphism between Riemann surfaces.

Definition

A function $F : M \to N$ is said to be a **biholormorphic** if it is a bijective and both $F : M \to N$ and $F^{-1} : N \to M$ are holomorphic.



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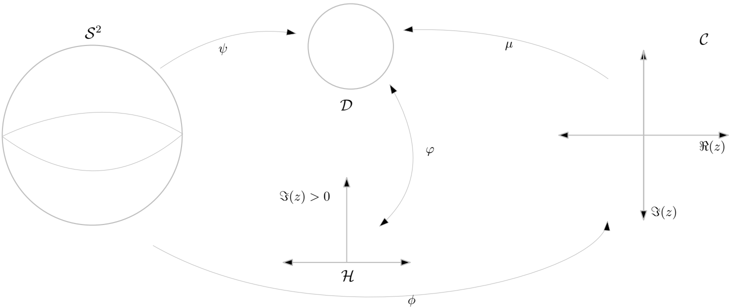
Are $\mathbb{C}$ $\widehat{\mathbb{C}}$ and $\mathbb{D}$ biholomorphic to each other?



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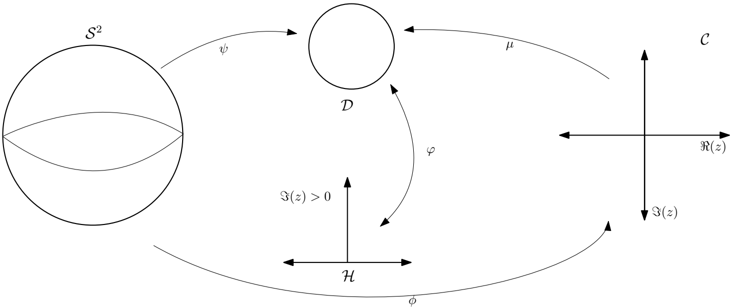

Figura: Likely biholomorphisms among $\mathbb{S}^2$, $\mathbb{D}$ and $\mathbb{C}$.

Theorem (Riemann Mapping Theorem)

Any non-empty simply connected domain $\Omega \subset \mathbb{C}$, which is not $\mathbb{C}$, is ***biholomorphic*** *to the unit disc $\mathbb{D}$.*

They aren't biholomorphic among them since:

- $\mu : \mathbb{C} \to \mathbb{D}$ neither by Liouville's theorem.
- $\psi : \mathbb{S}^2 \to \mathbb{D}$ and $\phi : \mathbb{S}^2 \to \mathbb{C}$ neither by compactness of $\mathbb{S}^2$.

However, $\mathbb{H}$ and $\mathbb{D}$ are biholomorphic via the following Möbius transformation

$$\varphi(z) = \frac{z - i}{z + i}$$

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Are there other Riemann surfaces beside $\mathbb{C}$, $\mathbb{D}$ and $\widehat{\mathbb{C}}$?

Theorem (The Uniformization Theorem (Poincaré, Koebe (1907)))

Every simply connected Riemann surface M is biholomorphic either to

- $\mathbb{D}$ *(hyperbolic),*
- $\mathbb{C}$ *(parabolic),*
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We know $\mathbb{D}$ and $\mathbb{C}$ aren't bilomorphic. Nonetheless, there exists a **diffeomorphism*** between them given

$$\phi(z) = \frac{z}{\sqrt{1 + \|z\|^2}}, \quad \phi^{-1}(w) = \frac{w}{\sqrt{1 - \|w\|^2}}$$

Example (*)

In $\mathbb{R}$, we can show

$$f : (\pi/2, \pi/2) \to \mathbb{R}$$
$$x \mapsto \tan(x)$$

is a diffeomorphism.

So, we point out the following observation:

- (Topological view) $M \cong N$ if there exists a homeomorphism $\phi : M \to N$ (topological invariant g, for instance).
- (Differential view) $M \cong N$ if there exists a **diffeomorphism** $\varphi : M \to N$.
- In general, every topological type splits into different diffeomorphy types. However, in the case of compact, orientable surface there is just one diffeomorphy type for every genus g.
- Be careful! in higher dimensions, **John Milnor** proved the topological space $\mathbb{S}^7$ admits 28 different differential structures.

Manifolds are used in other mathematical areas as well as physics. For instance,

- Algebraic geometry$\rightarrow$ elliptic curves on $\mathbb{C}$ (cryptography) $\leftarrow$ compact Riemann surfaces $g = 1$.
- Differential geometry$\rightarrow$ String Theory $\leftarrow$ physics.
- Topological quantum field theory $\longleftrightarrow$ Moduli spaces.

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TO BE CONTINUED...

THANK YOU!

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