Spherical Partition Schubert Varieties

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Abstract

A Schubert variety $X_{w\mathbf{B}}$ is called a partition Schubert variety (or a Ding's Schubert variety) if w is a 312-avoiding permutation. The complexity of an algebraic action $H \times X \to X$ is the codimension of a general orbit of H in X. We proved that every partition Schubert variety of torus-complexity one is spherical (for an appropriate reductive group action). We found the size of the family of such partition Schubert varieties by using Dyck paths.

Introduction

Spherical varieties. Let $\mathbf{G}$ be a complex connected reductive algebraic group. A normal $\mathbf{G}$-variety $\mathbf{Y}$ is called a spherical variety if it contains a dense orbit of some Borel subgroup $\mathbf{B} \subseteq \mathbf{G}$. Equivalently, $\mathbf{Y}$ is spherical if the codimension of a general $\mathbf{B}$-orbit is zero. Hence, the complexity, denoted $c_{\mathbf{B}}(\mathbf{Y})$, is zero. It is well-known that all *toric, flag, wonderful, and symmetric spaces* are examples of spherical varieties (see [10]).

Schubert varieties. Let $\mathbf{G} = \mathrm{GL}_n$, $\mathbf{B}$, and $\mathbf{T}$ be the general linear group over $\mathbb{C}$, the subgroup of upper triangular matrices, and the diagonal matrices in $\mathbf{B}$ respectively.

$$\mathbf{G}\,/\,\mathbf{B} = \bigsqcup_{w \in \mathfrak{S}_n} \mathbf{B}\,w\,\mathbf{B}\,/\,\mathbf{B}, \qquad \mathfrak{S}_n \cong \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\,\mathbf{T}\,.$$

A Schubert variety $X_{w\mathbf{B}}$ is the closure of a $\mathbf{B}$-orbit $\mathbf{B}\,w\,\mathbf{B}\,/\,\mathbf{B}$ in $\mathbf{G}\,/\,\mathbf{B}$. Schubert varieties are always normal (see [3]). The codimension of a general $\mathbf{T}$-orbit in $X_{w\mathbf{B}}$, denoted $c_{\mathbf{T}}(X_{w\mathbf{B}})$, is called the torus complexity of $X_{w\mathbf{B}}$. We are concerned about the situation where $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ and $X_{w\mathbf{B}}$ is spherical with respect to some reductive group action.

More precisely, there is a standard (w.r.t. $\mathbf{T}$) Levi subgroup $\mathbf{L} \subset \mathbf{G}$ acting on $X_{w\mathbf{B}}$. Hence, we have a Borel subgroup $\mathbf{B_L} \supset \mathbf{T}$ of $\mathbf{L}$ acting on $X_{w\mathbf{B}}$. We head towards *under what conditions* $c_{\mathbf{B_L}}(X_{w\mathbf{B}}) = 0$*?*

Bruhat Order

Weyl group. Since $(\mathfrak{S}_n, S)$ is a Coxeter system where $S = \{s_1, \dots, s_{n-1}\}$ and $s_i = (i \quad i+1)$, every w in $\mathfrak{S}_n$ can be written as product of the s_i's. If $w = s_{i_1} \cdots s_{i_\ell}$ and ℓ is minimal among all such expressions, then $\ell := \ell(w)$ is said to be the length of w, and the expression $s_{i_1} \cdots s_{i_\ell}$ is called a reduced decomposition for w.

Let $\mathfrak{S}_I$ be the parabolic subgroup of $\mathfrak{S}_n$ generated by $I \subseteq S$ and $w_0(I)$ its longest element. A standard Coxeter element $c \in \mathfrak{S}_I$ is any product of the elements of I listed in some order. Denote $J(w) := \{s \in S : \ell(sw) < \ell(w)\}$ the left descent set of $w = w_1 w_2 \cdots w_n$.

Bruhat–Chevalley order. Let $T := \{usu^{-1} : s \in S, \quad u \in \mathfrak{S}_n\}$ and w, v in $\mathfrak{S}_n$. The partial order $\leq$, defined by $v \leq w$, if and only if

$$w = vt \text{ for some } t \in T, \qquad \ell(w) = \ell(v) + 1$$

is the Bruhat order on $\mathfrak{S}_n$ which is a graded poset with rank function ℓ (see [2]). Likewise, the Bruhat–Chevalley order $(\mathfrak{S}_n, \leq)$ is defined by

$$v \leq w \iff X_{v\mathbf{B}} \subseteq X_{w\mathbf{B}}, \qquad \ell(w) = \dim X_{w\mathbf{B}}.$$

Fig. 1 depicts the Bruhat order on $\mathfrak{S}_4$ in terms of reduced words.

Pattern Avoidance

Pattern Avoidance. For $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$ with $k \leq n$, we say the permutation w contains the pattern p if there exits a sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that $w(i_1) \cdots w(i_k)$ is in the same relative order as $p(1) \cdots p(k)$. If w does not contain p, then w is said to avoid p.

Partition Schubert Varieties. We call $X_{w\mathbf{B}}$ a partition Schubert variety, denoted by $\mathscr{D}_w$, if w is a 312-avoiding permutation. (It is also called a Ding's Schubert variety in [5].) It follows from [8] that all partition Schubert varieties are smooth.

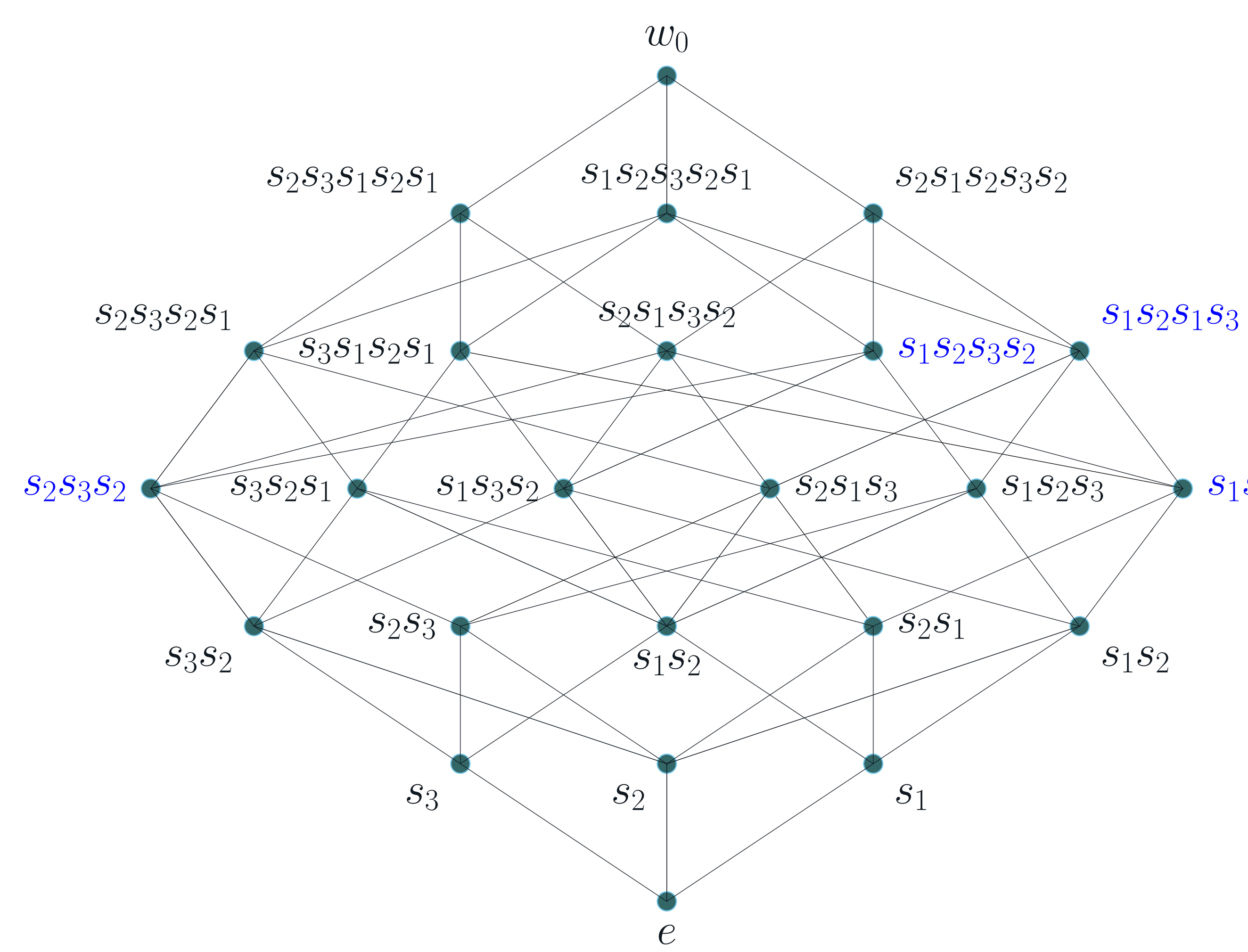

Fig. 1: Bruhat order for $\mathfrak{S}_4$.

Classification

Lee-Masuda-Park [9]. $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ and smooth $\iff$ w contains the pattern 321 exactly once and avoids 3412 $\iff$ there exists a reduced word of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions. Moreover, $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ and singular $\iff$ w contains the pattern 3412 exactly once and avoids the pattern 321.

Gao-Hodges-Yong, Gaetz [6, 7]. $c_{\mathbf{B_L}}(X_{w\mathbf{B}}) = 0 \iff w_0(J(w))w$ is a Coxeter element of $W_{J(w)}$ $\iff$ w avoids the following 21 patterns

$$\mathscr{P} := \begin{Bmatrix} 24531 & 25314 & 25341 & 34512 & 34521 & 35412 & 35421 \\ 42531 & 45123 & 45213 & 45231 & 45312 & 52314 & 52341 \\ 53124 & 53142 & 53412 & 53421 & 54123 & 54213 & 54231 \end{Bmatrix}$$

Result 1(Can-D). Let $\mathscr{D}_w \subset \mathrm{GL}_n\,/\,\mathbf{B}$ be a partition Schubert variety such that $c_{\mathbf{T}}(\mathscr{D}_w) = 1$. Then $\mathscr{D}_w$ is a spherical $\mathbf{L}$-variety, where $\mathbf{L}$ is a Levi factor of the stabilizer of $\mathscr{D}_w$ in GL_n. We denote $\mathscr{D}(n)$ the set of all such partition Schubert varieties.

Proof. $c_{\mathbf{T}}(\mathscr{D}_w) = 1 \iff w$ contains exactly once the pattern 321. Then check off the 21 patterns in $\mathscr{P}$. □

Result 2(Can-D). All singular Schubert varieties of $\mathbf{T}$-complexity one are spherical.

Example 1. Let $\mathscr{D}_w$ be of complexity one in $\mathrm{GL}_5(\mathbb{C})/\,\mathbf{B}_5(\mathbb{C})$. Then w is an element of

$$\mathscr{D}(5) = \begin{Bmatrix} 12543 & 13542 & 14325 & 14352 & 21543 & 23541 \\ 24315 & 24351 & 32145 & 32154 & 32415 & 32451 \end{Bmatrix}$$

Dyck Paths

A Pretty Construction. By [1], we know that there is a bijection

$$\mathscr{D}(n) \xrightarrow{\ \psi\ } L^+_{n,n}$$
$$\ell(w) \longmapsto \mathrm{area}(\psi(w)) := \pi$$

Result 3(Can-D). Let $\pi \in L^+_{n,n}$ and $w = w_1 w_2 \cdots w_n$ be such that $\phi(\pi) = w$ for $n \geq 4$. Then w has a unique 321 $\iff$ π has a unique peak at the second diagonal and no other peaks at the r-th diagonal for $r \geq 3$.

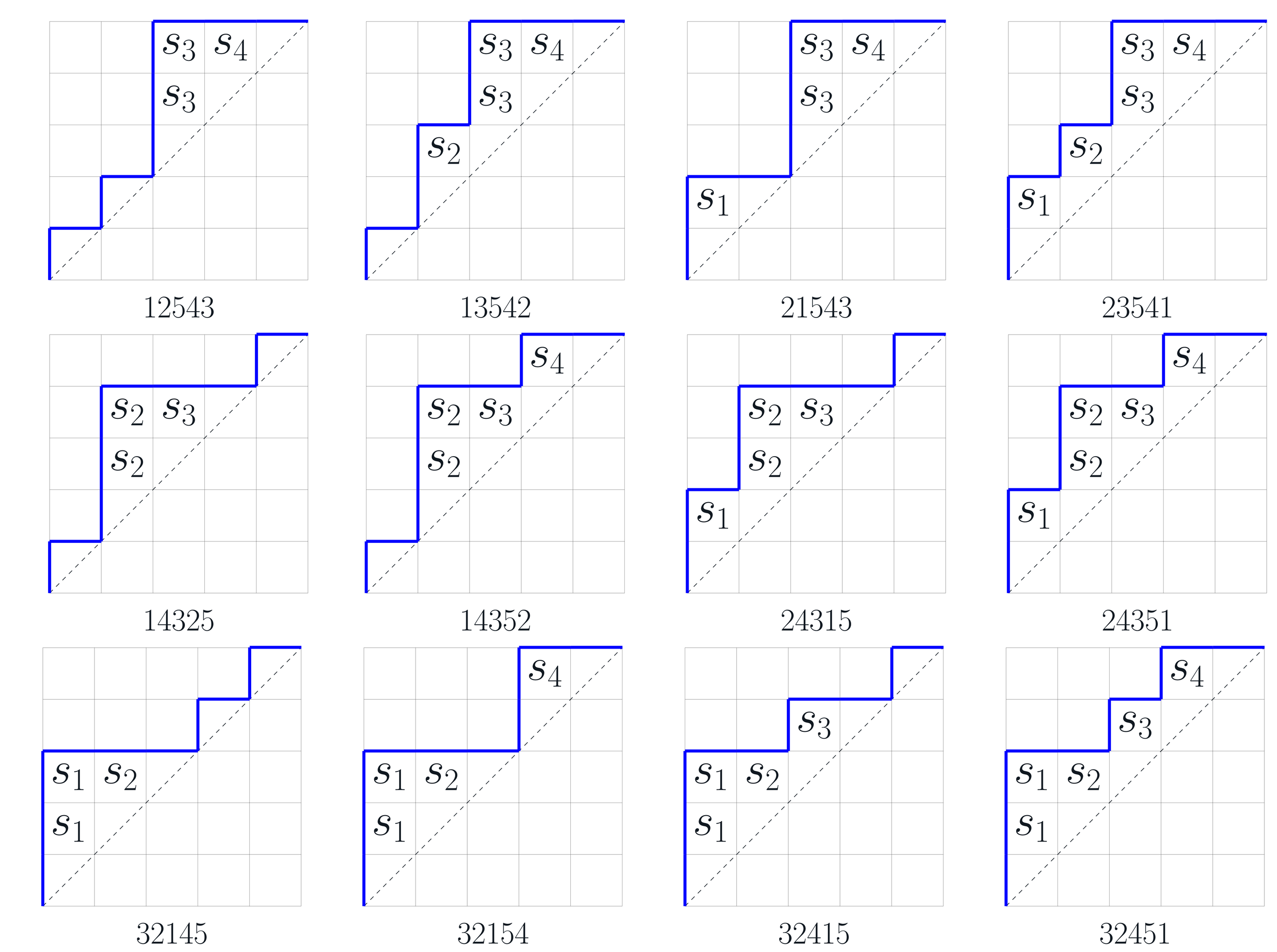

Fig. 2: The Dyck paths of $\mathscr{D}(5)$

Result 4 (Can-D). If $n \geq 4$, the cardinality of $\mathscr{D}(n)$ is given by $2^{n-3}(n-2)$. For example, Fig. 2 depicts all 12 elements of $\mathscr{D}(5)$.

More surprises. More cute results and future work can be found in [4].

References

[1] J. Bandlow and K. Killpatrick. "An area-to-inv bijection between Dyck paths and 312-avoiding permutations". In: *Electron. J. Combin.* 8.1 (2001), Research Paper 40, 16.

[2] A. Björner and F. Brenti. *Combinatorics of Coxeter groups.* Graduate Texts in Mathematics. Springer, 2005.

[3] M. Brion. "Lectures on the geometry of flag varieties". In: *Topics in cohomological studies of algebraic varieties* 19.1 (2005), pp. 33–85.

[4] M. B. Can and N. Diaz Morera. *Nearly Toric Schubert Varieties and Dyck Paths.* 2022. URL: `https://arxiv.org/abs/2212.01234`.

[5] M. Develin, J. L. Martin, and V. Reiner. "Classification of Ding's Schubert Varieties: Finer Rook Equivalence". In: *Canadian Journal of Mathematics* 59.1 (2007), pp. 36–62.

[6] C. Gaetz. "Spherical Schubert varieties and pattern avoidance". In: *Selecta Math. (N.S.)* 28.2 (2022), Paper No. 44, 9.

[7] Y. Gao, R. Hodges, and A. Yong. "Classifying Levi-spherical Schubert varieties". In: *Sém. Lothar. Combin.* 86B (2022), Art. 29, 12.

[8] V. Lakshmibai and B. Sandhya. "Criterion for smoothness of Schubert varieties in Sl(n)/B". In: *Proc. Indian Acad. Sci. Math. Sci.* 100.1 (1990), pp. 45–52.

[9] E. Lee, M. Masuda, and S. Park. "On Schubert varieties of complexity one". In: *Pacific J. Math.* 315.2 (2021), pp. 419–447.

[10] D. A. Timashev. *Homogeneous spaces and equivariant embeddings.* Vol. 138. Encyclopaedia of Mathematical Sciences. Invariant Theory and Algebraic Transformation Groups, 8. Springer, Heidelberg, 2011.