Dyck paths and spherical partition Schubert varieties

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Abstract

A Schubert variety $X_{w\mathbf{B}}$ is called a partition Schubert variety (or a Ding's Schubert variety) if w is a 312-avoiding permutation. The complexity of an algebraic action $H \times X \to X$ is the codimension of a general orbit of H in X. We proved that every partition Schubert variety of torus-complexity one is spherical (for an appropriate reductive group action). We found the size of the family of such partition Schubert varieties by using Dyck paths. This work is based on [4].

Introduction

Spherical varieties. Let $\mathbf{G}$ be a complex connected reductive algebraic group. A normal $\mathbf{G}$-variety $\mathbf{Y}$ is called a spherical variety if it contains a dense orbit of some Borel subgroup $\mathbf{B} \subseteq \mathbf{G}$. Equivalently, $\mathbf{Y}$ is spherical if the codimension of a general $\mathbf{B}$-orbit is zero. Hence, the complexity, denoted $c_{\mathbf{B}}(\mathbf{Y})$, is zero. It is well-known that all *toric, flag, wonderful, and symmetric spaces* are examples of spherical varieties (see [12]).

Schubert varieties. Let $\mathbf{G} = \mathrm{GL}_n$, $\mathbf{B}$, and $\mathbf{T}$ be the general linear group over $\mathbb{C}$, the subgroup of upper triangular matrices, and the diagonal matrices in $\mathbf{B}$ respectively.

$$\mathbf{G}\,/\,\mathbf{B} = \bigsqcup_{w\in\mathfrak{S}_n} \mathbf{B}\,w\,\mathbf{B}\,/\,\mathbf{B}, \qquad \mathfrak{S}_n \cong \mathrm{N}_{\mathbf{G}}(\mathbf{T})/\,\mathbf{T}\,.$$

A Schubert variety $X_{w\mathbf{B}}$ is the closure of a $\mathbf{B}$-orbit $\mathbf{B}\,w\,\mathbf{B}\,/\,\mathbf{B}$ in $\mathbf{G}\,/\,\mathbf{B}$. Schubert varieties are always normal (see [3]). The codimension of a general $\mathbf{T}$-orbit in $X_{w\mathbf{B}}$, denoted $c_{\mathbf{T}}(X_{w\mathbf{B}})$, is called the torus complexity of $X_{w\mathbf{B}}$.

Spherical Schubert varieties. Let $\mathbf{L}$ denote the Levi subgroup of the isotropic subgroup of $X_{w\mathbf{B}}$ in $\mathbf{G}$. Hence, we have a Borel subgroup $\mathbf{B_L} \supset \mathbf{T}$ of $\mathbf{L}$ acting on $X_{w\mathbf{B}}$ (see [11]). If $\mathbf{B_L}$ has only finitely many orbits in $X_{w\mathbf{B}}$, then we call $X_{w\mathbf{B}}$ a spherical Schubert variety.

If $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$, when does $c_{\mathbf{B_L}}(X_{w\mathbf{B}}) = 0$ hold?

Bruhat Order

Weyl group. Since $(\mathfrak{S}_n, S)$ is a Coxeter system where $S = \{s_1, \ldots, s_{n-1}\}$ and $s_i = (i \;\; i+1)$, every w in $\mathfrak{S}_n$ can be written as product of the s_i's. If $w = s_{i_1}\cdots s_{i_\ell}$ and ℓ is minimal among all such expressions, then $\ell := \ell(w)$ is said to be the length of w, and the expression $s_{i_1}\cdots s_{i_\ell}$ is called a reduced decomposition for w.

Let $\mathfrak{S}_I$ be the parabolic subgroup of $\mathfrak{S}_n$ generated by $I \subseteq S$ and $w_0(I)$ its longest element. A standard Coxeter element $c \in \mathfrak{S}_I$ is any product of the elements of I listed in some order. Denote $J(w) := \{s \in S : \ell(sw) < \ell(w)\}$ the left descent set of $w = w_1 w_2 \cdots w_n$.

Bruhat–Chevalley order. Let $T := \{usu^{-1} : s \in S, \quad u \in \mathfrak{S}_n\}$ and w, v in $\mathfrak{S}_n$. The partial order $\leq$, defined by $v \leq w$, if and only if

$$w = vt \text{ for some } t \in T, \qquad \ell(w) = \ell(v) + 1$$

is the Bruhat order on $\mathfrak{S}_n$ which is a graded poset with rank function ℓ (see [2]). Likewise, the Bruhat–Chevalley order $(\mathfrak{S}_n, \leq)$ is defined by

$$v \leq w \iff X_{v\mathbf{B}} \subseteq X_{w\mathbf{B}}, \qquad \ell(w) = \dim X_{w\mathbf{B}}.$$

Pattern Avoidance

Pattern Avoidance. For $w \in \mathfrak{S}_n$ and $p \in \mathfrak{S}_k$ with $k \leq n$, we say the permutation w contains the pattern p if there exits a sequence $1 \leq i_1 < \cdots < i_k \leq n$ such that $w(i_1)\cdots w(i_k)$ is in the same relative order as $p(1)\cdots p(k)$. If w does not contain p, then w is said to avoid p.

Partition Schubert Varieties. We call $X_{w\mathbf{B}}$ a partition Schubert variety, denoted by $\mathscr{D}_w$, if w is a 312-avoiding permutation. (It is also called a Ding's Schubert variety in [5].) It follows from [8] that all partition Schubert varieties are smooth.

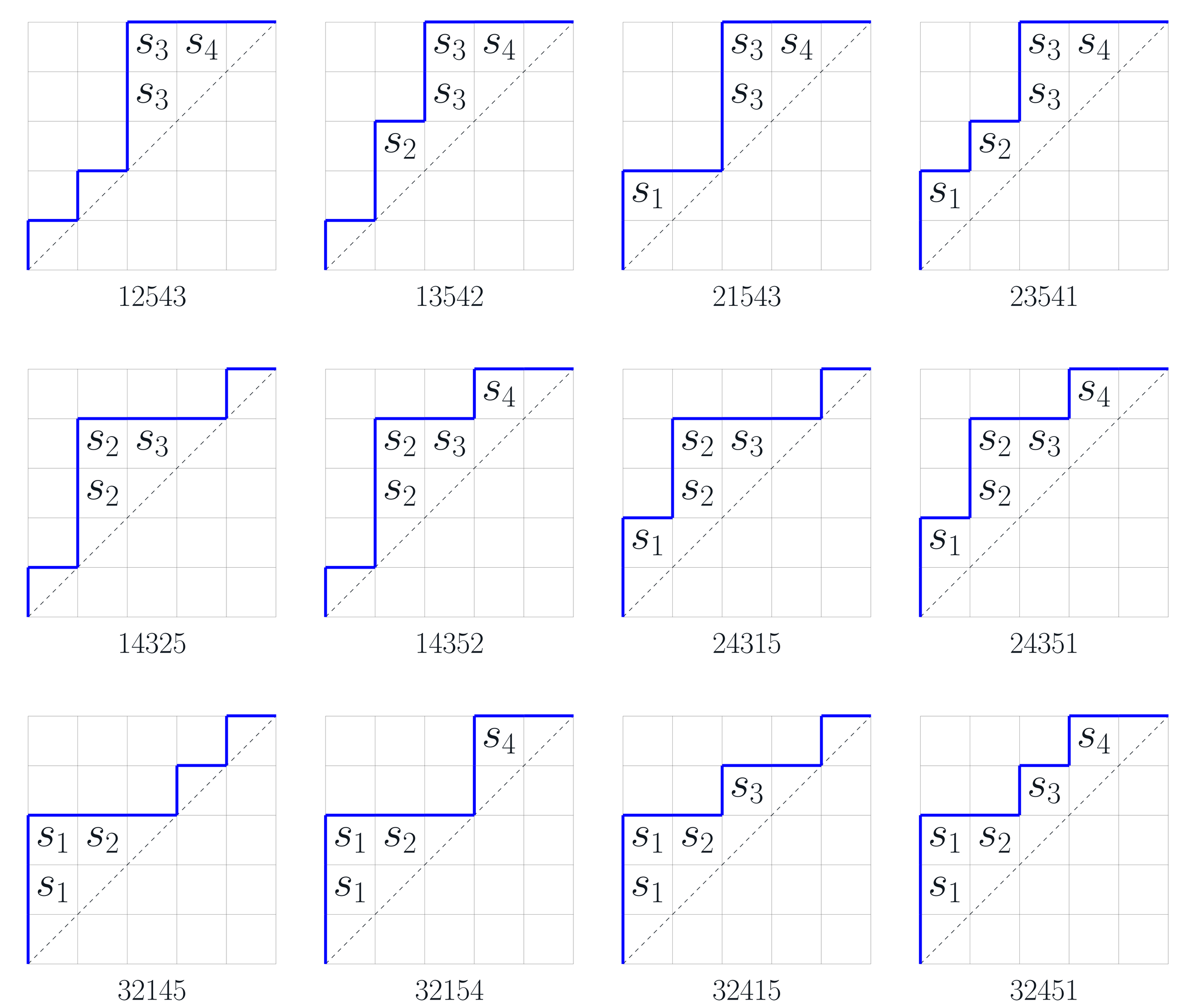


Fig. 1: Dyck paths of $\mathbf{NT}_5^{312}$.

Classification

Lee-Masuda-Park [9]. $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ and smooth $\iff$ w contains the pattern 321 exactly once and avoids 3412 $\iff$ there exists a reduced word of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions. Moreover, $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ and singular $\iff$ w contains the pattern 3412 exactly once and avoids the pattern 321.

Gao-Hodges-Yong, Gaetz [6, 7]. $c_{\mathbf{B_L}}(X_{w\mathbf{B}}) = 0 \iff w_0(J(w))w$ is a Coxeter element of $\mathfrak{S}_{J(w)} \iff w$ avoids the following 21 patterns

$$\mathscr{P} := \left\{\begin{matrix} 24531 & 25314 & 25341 & 34512 & 34521 & 35412 & 35421 \\ 42531 & 45123 & 45213 & 45231 & 45312 & 52314 & 52341 \\ 53124 & 53142 & 53412 & 53421 & 54123 & 54213 & 54231 \end{matrix}\right\}$$

Theorem 1 (Can-D). Let $\mathscr{D}_w \subset \mathrm{GL}_n\,/\,\mathbf{B}$ be a partition Schubert variety such that $c_{\mathbf{T}}(\mathscr{D}_w) = 1$. Then $\mathscr{D}_w$ is a spherical $\mathbf{L}$-variety, where $\mathbf{L}$ is a Levi factor of the stabilizer of $\mathscr{D}_w$ in GL_n. We denote $\mathbf{NT}_n^{312}$ the set of spherical partition Schubert varieties with complexity one.

Corollary 1(Can-D). All singular Schubert varieties of $\mathbf{T}$-complexity one are spherical.

Example 1. In the flag $\mathrm{GL}_5(\mathbb{C})/\,\mathbf{B}_5(\mathbb{C})$, w takes the place of

$$\mathbf{NT}_5^{312} = \left\{\begin{matrix} 12543 & 13542 & 14325 & 14352 & 21543 & 23541 \\ 24315 & 24351 & 32145 & 32154 & 32415 & 32451 \end{matrix}\right\}$$

Dyck Paths

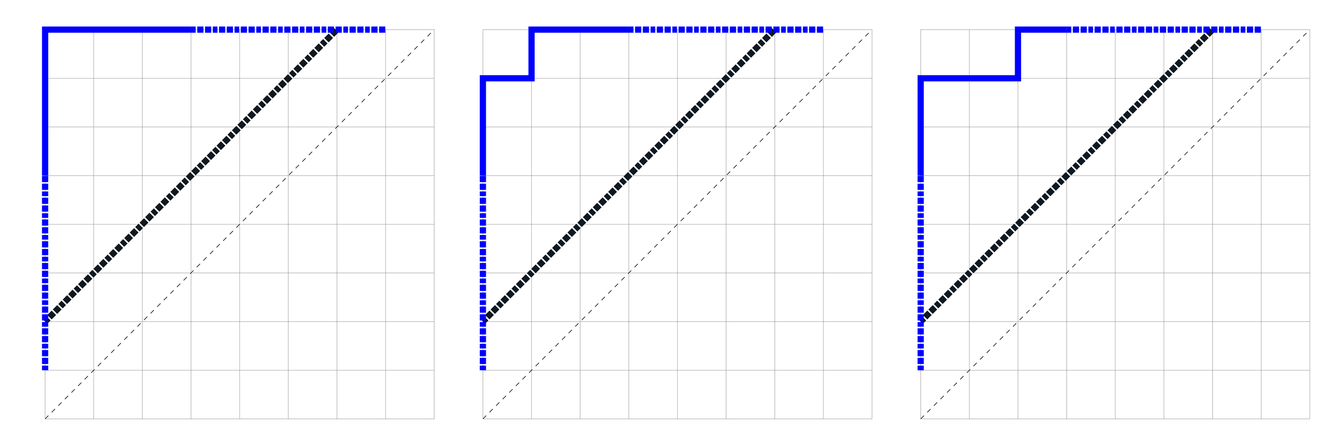
Bandlow-Killpatrick isomorphism [1].

$$\mathfrak{S}_n^{312} \underset{\phi}{\overset{\psi}{\rightleftarrows}} \mathscr{L}_{n,n}^{+}\;;\;\; \ell(w) \longmapsto \mathrm{area}(\psi(w)) := \pi$$

Theorem 2 (Can-D). Let $\pi \in \mathscr{L}_{n,n}^{+}$ and $w = w_1 w_2 \cdots w_n$ be such that $\phi(\pi) = w$ for $n \geq 4$. Then w has a unique 321 $\iff$ π has a unique peak at the second diagonal and no other peaks at the r-th diagonal for $r \geq 3$.

Corollary 2 (Can-D). $|\mathbf{NT}_n^{312}| = 2^{n-3}(n-2)$. For example, Fig. 1 depicts all 12 elements of $\mathbf{NT}_5^{312}$.

Theorem 3 (Can-D). Let $\pi \in \mathscr{L}_{n,n}^{+}$ and $w \in \mathfrak{S}_n^{312}$ be such that $\phi(\pi) = w$. Then $X_{w\mathbf{B}}$ is a spherical Schubert variety $\iff \pi^{(2)} = \emptyset$, or every connected component M of $\pi^{(2)}$ is either an elbow, dimple, or ledge as depicted below



Shortcoming and upcoming work. If $w = 25314$, we found out that $c_{\mathbf{T}}(X_{w\mathbf{B}}) = 1$ is smooth, yet $c_{\mathbf{B_L}}(X_{w\mathbf{B}}) \neq 0$. By using [10], we discovered that

n	1	2	3	4	5	6	7	8	9
r_n	0	0	1	6	24	84	275	864	2639

$$\rightsquigarrow r_{n+2} = n \cdot \mathscr{F}_{2n}, \quad n \geq 0.$$

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