Lexicographic shellability of Sects

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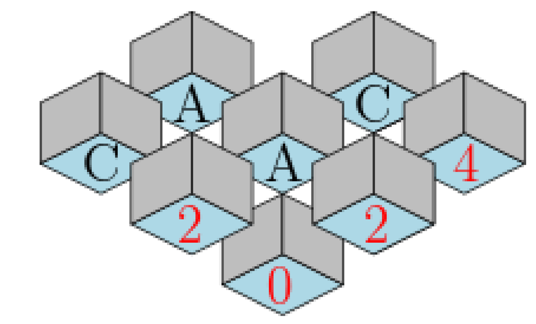


Abstract

In this investigation, based on [2], we show that the Bruhat order on the **sects** of a *symmetric variety of type AIII* are **lexicographically shellable**. Our proof proceeds from a description of these posets as rook placements in a partition shape which fits in a $p \times q$ rectangle. This allows us to extend an *EL*-labeling of the *rook monoid* given by Can [3] to an arbitrary sect. As a special case, our result implies that the Bruhat order on *matrix Schubert* varieties is lexicographically shellable.

Clans, Sects, and Rooks

Definition 1. Let p and q be two positive integers such that $p+q=n$. A (p,q)**-clan** is an ordered set of n symbols $c_1 \dots c_n$ such that:

- Each symbol c_i is either "+", "−", or a natural number.
- If $c_i \in \mathbb{N}$, then there is a unique index $j \neq i$ such that $c_i = c_j$.
- The difference between the numbers of "+" and "−" symbols in the clan is equal to $p-q$. If $q > p$, then we have $q-p$ more minus signs than plus signs.

E.g. $\gamma_1 = +1212-$ and $\gamma_2 = +1717-$ are equivalent $(3,3)$-clans. The set of all (p,q)-clans is denoted by $\mathcal{C}_{p,q}$.

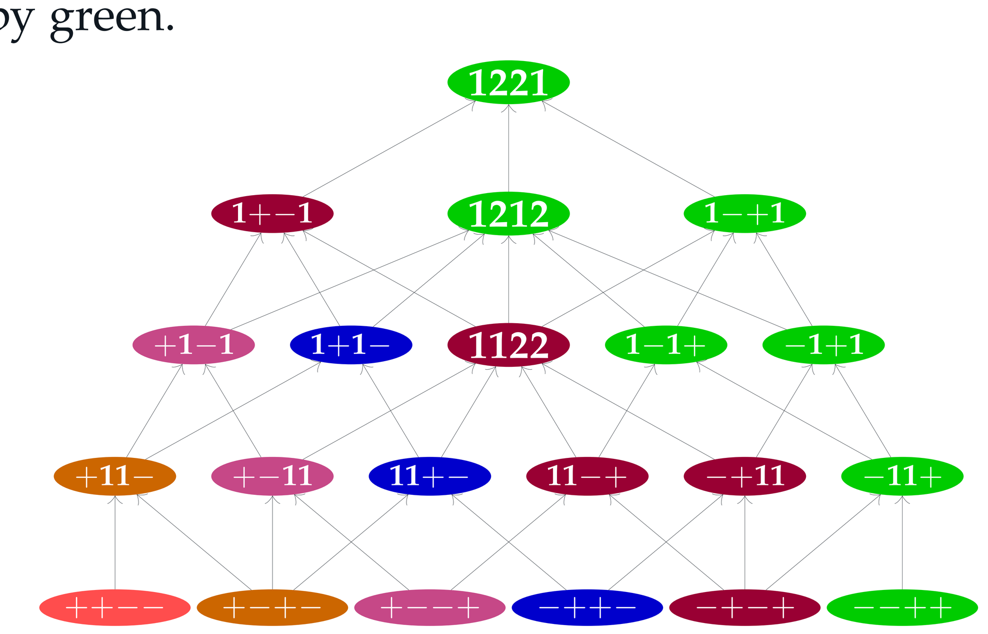
Matsuki-Oshima-Yamamoto [5, 7]. The orbits of $\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C}) := \mathbf{L}$ on $\mathrm{GL}_{p+q}(\mathbb{C})/\mathbf{B}$, where $\mathbf{B}$ is the *Borel* subgroup of $\mathrm{GL}_{p+q}(\mathbb{C}) := \mathbf{G}$, are parameterized by $\mathcal{C}_{p,q}$. The *Bruhat poset* $(\mathcal{C}_{p,q}, \leq)$ is defined by

$$\gamma_1 \leq \gamma_2 \iff \mathcal{O}_{\gamma_1} \subseteq \overline{\mathcal{O}_{\gamma_2}}$$

for γ_1, γ_2 in $\mathcal{C}_{p,q}$ corresponding to $\mathbf{B}$-orbits $\mathcal{O}_{\gamma_1}$ and $\mathcal{O}_{\gamma_2}$ respectively. The **matchless (base clan)** clans τ_γ, consisting only "+" and "−", correspond to min-dimensional orbits.

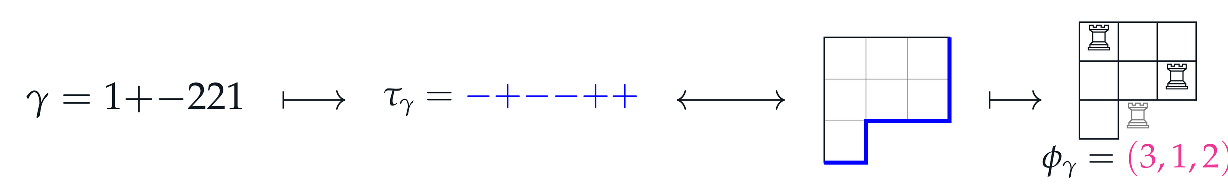
Definition 2. Let C_λ denote the *Schubert cell* of $\mathbf{G}/\mathbf{P}$ associated to the partition $\lambda \in \binom{[p+q]}{p}$ where $\mathbf{P}$ is the *parabolic* subgroup. Then the **sect** $\mathcal{C}^\lambda_{p,q}$ is the collection of clans γ whose corresponding orbits satisfy $\pi(\mathcal{O}_\gamma) = C_\lambda$ where $\pi : \mathbf{G}/\mathbf{L} \to \mathbf{G}/\mathbf{P}$ is the natural projection map.

E.g.1. The Bruhat poset of $\mathcal{C}_{2,2}$ can be depicted below by using Wyser's characterization [6]. Also, the sect $\mathcal{C}^\lambda_{2,2}$ for $\lambda = (2,2)$ is colored by green.



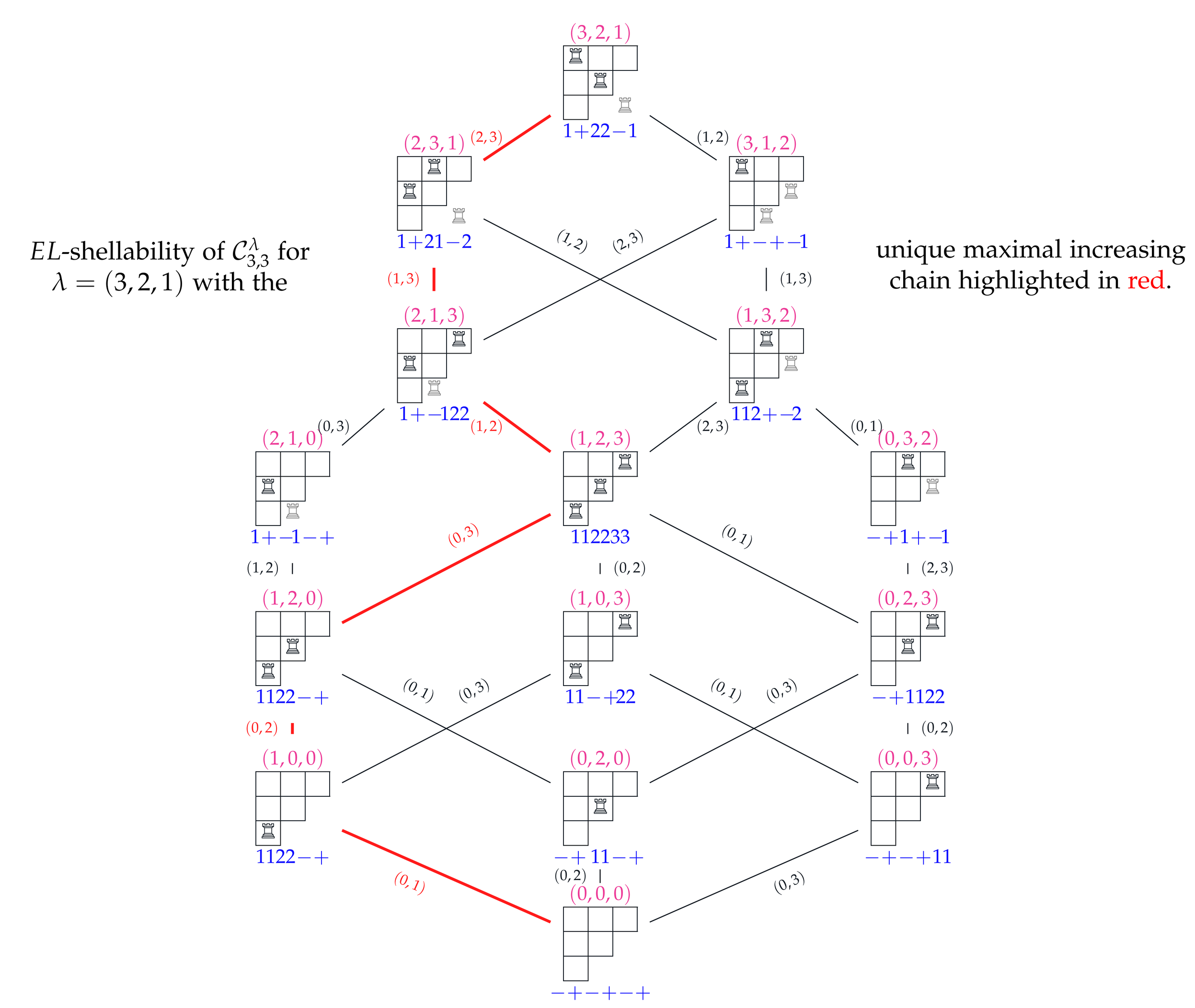
Bingham-Can[1]. Matchless clans are in one-to-one correspondence with sects by drawing a lattice path from the origin to the point (q,p) where "−" correspond to "east" steps and "+" to "north" steps. The partition λ then consists of the boxes that lie inside the rectangle and weakly above this lattice path.

Definition 3. Let $R(\lambda)$ denote the set of **rook placements** of partition λ. For ρ, π in $R(\lambda)$, we say $\rho \leq \pi \iff rt_\rho \leq rt_\pi$ where $R(\lambda) \xrightarrow{rt} \mathbb{N}$ is a labeling of the boxes of $[\lambda]$ by the number of rooks weakly northwest of each box.



Bingham-Can[1]. There is an isomorphism between the posets $R(\lambda)$ and $\mathcal{C}^\lambda_{p,q}$ as follows

- Bring down the clan $\gamma = c_1 \cdots c_{p+q}$ to the base clan τ_γ by replacing c_i by a "−" and c_j by a "+" whenever $c_i = c_j \in \mathbb{N}$ with $i < j$ for every i, j.
- Let the positions of the − symbols in τ_γ be $i_1, \dots, i_q$ and the positions of the + symbols be $j_1, \dots, j_p$ from left to right.
- For each pair $c_{i_k} = c_{j_l} \in \mathbb{N}$ in γ, we place a rook in the square with northeast corner (k,l).



EL-shellability of $\mathcal{C}^\lambda_{3,3}$ for $\lambda = (3,2,1)$ with the unique maximal increasing chain highlighted in red.

Definition 4. The **partial permutation** associated to a clan $\gamma = c_1 \cdots c_{p+q}$ in $\mathcal{C}_{p,q}$ is the function $\phi_\gamma : [q] \to [p] \cup \{0\}$ defined algorithmically as follows. Label the positions of the "−" in τ_γ by $i_1, \dots, i_q$ and the positions of the "+" as $j_1, \dots, j_p$ in ascending order. Then we read the symbols $c_1 \cdots c_{p+q}$ left to right and construct ϕ_γ as follows.

1. If $c_{i_s} = c_{j_t} \in \mathbb{N}$ in γ, then $\phi_\gamma(s) = t$. These are the rooks that are placed within the associated partition λ under B-C [1].
2. After the previous step, we modify γ by iteratively replacing all 1212 patterns by 1221 patterns to obtain a clan which we call $\hat{\gamma}_0 \in \mathcal{C}^\lambda_{p,q}$ and which has symbols $\hat{c}_1 \cdots \hat{c}_{p+q}$.
3. For each simple, innermost $1{+}{-}1$ pattern in $\hat{\gamma}_0$ of the form $\hat{c}_a \hat{c}_{j_l} \hat{c}_{i_k} \hat{c}_b$, set $\phi_\gamma(k) = l$. Then delete all of the symbols involved in any simple, innermost $1{+}{-}1$ pattern to obtain a new clan $\hat{\gamma}_1$ which inherits position labels from $\hat{\gamma}_0$.
4. Repeat the procedure of the previous step on $\hat{\gamma}_1$, and so on until we obtain a clan $\hat{\gamma}_s$ which is free of $1{+}{-}1$ patterns.
5. Once the clan $\hat{\gamma}_s$ which is free of $1{+}{-}1$ patterns is obtained, for any $k \in [q]$ which has not yet been assigned we let $\phi_\gamma(k) = 0$.

We will represent ϕ_γ using one-line notation. For instance, the partial permutation associated to the clan $\gamma = 1{+}{-}221$ is $\phi_\gamma = (3,1,2)$.

Shelling

Definition 5. Let $C(\mathcal{P}) := \{(u,v) \in \mathcal{P} \times \mathcal{P} \mid u \lessdot v\}$ denote the set of *covering relations* in a poset $\mathcal{P}$. An *EL*-**labelling** on $(\mathcal{P}, <)$ is a map $\eta : C(\mathcal{P}) \to (\Lambda, \leq_{\mathbf{LEX}})$ holding the following:

- For each $u < v$, there is a unique saturated chain $u \lessdot u_1 \lessdot \cdots \lessdot u_k \lessdot v$ with $\eta(u,u_1) \leq \eta(u_1,u_2) \leq \cdots \leq \eta(u_k, v)$.
- The above label sequence is lexicographically *smaller* than the label sequence for every other saturated chain from u to v.

E.g.2. Let $\mathcal{R}_n$ denote the *rook monoid*. Choose $x = (a_1, \dots, a_n), y = (a_1, \dots, a_n) \in \mathcal{R}_n$.

- (*Type 1*) Assume that $a_k = b_k$ for all $k = \{1, \dots, \hat{i}, \dots, n\}$ and that $a_i < b_i$. Then, y covers x if and only if either
  - ⋆ $0 = a_i$, and there exits a sequence $\{1 \leq j_1 < \cdots < j_s < i\}$ such that $a_{j_1}, \dots, a_{j_s}$ is equal to $\{1, \dots, s\}$, $b_i = s+1$, and $b_j = a_j > 0$ for $j > i$, or
  - ⋆ $0 < a_i$, and there exits a sequence $1 \leq j_1 < \cdots < j_s < i$ such that $\{a_{j_1}, \dots, a_{j_s}\}$ is equal to $\{a_i+1, \dots, a_i+s\}$, $b_i = s+1$, and $b_i = a_i + s + 1 > 0$.
- (*Type 2*) Suppose that $a_j = b_i, a_i = b_j$, and $a_j < a_i$ for $i < j$. Moreover, suppose that for all $k \in \{1, \dots, \hat{i}, \dots, \hat{j}, \dots, n\}$, $a_k = b_k$. Then, $x \lessdot y$ if and only if either $a_j < a_s$, or $a_s < a_i$ for $s = i+1, \dots, j-1$.

Mahir proved in [3] that $\mathcal{R}_n$ is lexicographic shellable by deploying

$$C(\mathcal{R}_n) \xrightarrow{\quad \eta \quad} \Lambda := [n] \times [n]$$

$$(x,y) \longmapsto \eta(x,y) := \begin{cases} (a_i, b_i) \text{ if } x \lessdot y \text{ by type 1} \\ (a_i, a_j) \text{ if } x \lessdot y \text{ by type 2} \end{cases}$$

Bingham-Diaz [2]. The restriction of Bruhat order on $\mathrm{GL}_{p+q}/(\mathrm{GL}_p \times \mathrm{GL}_q)$ to any sect $\mathcal{C}^\lambda_{p,q}$ gives an *EL*-shellable poset. In particular, the Bruhat order on matrix Schubert varieties is *EL*-shellable.

The *coverings* and *labeling* for $\mathcal{C}^\lambda_{p,q}$ are rooted in the concepts developed by Incitti, Wyser, and Can in their works [4, 6, 3]. For exact details, please refer to our in-depth exploration in [2]...

References

[1] Aram Bingham and Mahir Bilen Can. "Sects". In: *J. Algebra* 560 (2020), pp. 192–218.

[2] Aram Bingham and Nestor Diaz Morera. "Lexicographic shellability of sects". In: (2023). arXiv: `2312.15093 [math.CO]`.

[3] Mahir Bilen Can. "The rook monoid is lexicographically shellable". In: *European J. Combin.* 81 (2019), pp. 265–275.

[4] Federico Incitti. "The Bruhat order on the involutions of the symmetric group". In: *J. Algebraic Combin.* 20.3 (2004), pp. 243–261.

[5] Toshihiko Matsuki and Toshio Ōshima. "Embeddings of discrete series into principal series". In: vol. 82. Progr. Math. 1990, pp. 147–175.

[6] Benjamin J. Wyser. "The Bruhat order on clans". In: *J. Algebraic Combin.* 44.3 (2016), pp. 495–517.

[7] Atsuko Yamamoto. "Orbits in the flag variety and images of the moment map for classical groups. I". In: *Represent. Theory* 1 (1997).