Ding and Schubert Varieties

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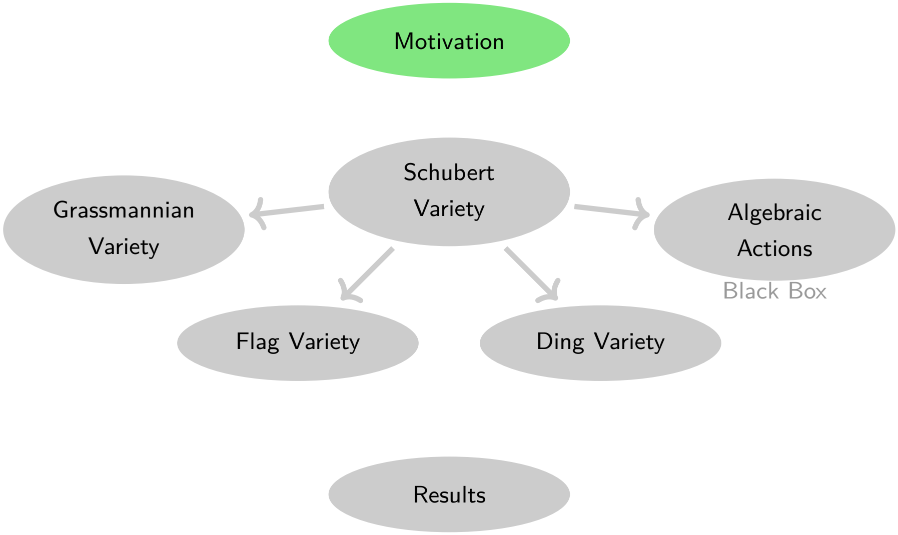
Outline



Motivation
Schubert Variety
Grassmannian Variety
Algebraic Actions
Black Box
Flag Variety
Ding Variety
Results

The Stem $\rightsquigarrow$ combinatorics

- A partition of a number n is a *non-increasing sequence* of non-negative integer $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d)$ such that

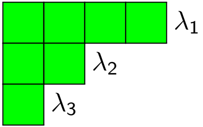
$$n = \sum \lambda_i := |\lambda|, \quad \ell(\lambda) := |\lambda_i : \lambda_i \neq 0| \rightsquigarrow \text{length}$$

  - e.g. $7 = \cdots = 4 + 2 + 1 = \cdots , \quad \lambda = (4, 2, 1) \rightsquigarrow \ell(\lambda) = 3$

- The diagram of λ

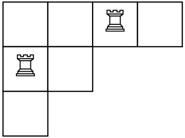
$$\mathrm{dg}(\lambda) := \{(i, j) \in \mathbb{N}^2 \mid 1 \leq i \leq d, \quad 1 \leq j \leq \lambda_i\}$$

- The Ferrer board F_λ of a partition λ is the left-aligned partial grid of boxes in which the i^{th} row from the top has λ_i boxes.
  - e.g.



$$\lambda = (4, 2, 1) = (\lambda_1, \lambda_2, \lambda_3)$$

- A placement of d rooks on a given F_λ is a subset of d squares in F_λ. A non-attacking placement of d rooks on F_λ is a subset of d squares in F_λ such no two squares lie in the same row or column.



Non-attacking placement on F_λ

- Let $r_d(\lambda)$ be the number of non-attacking placements of d rooks on F_λ. The rook polynomial of λ is

$$R_\lambda(x) := \sum_{d=0}^{n} r_d(\lambda)x^k$$

- Two partiions λ and ν are rook-equivalence if the have the same rook polynomial i.e., $r_d(\lambda) = r_d(\nu)$ for all $d \geq 0$.

- Since $R_{(2,2)}(x) = 2x^2 + 4x + 1 = R_{(3,1)}(x) = R_{(2,1,1)}(x)$,
  - *Kaplansky & Riordan (1946)* considered the problem of when F_λ and F_ν are rook-equivalent.
- *Garcia & Remmel (1984)* defined a q-rook polynomial $R_d(F_\lambda, q)$ that q-count the d-rook placements on F_λ by a certain *inversion* statistic generalizing **inversions** of permutations.
  - They showed q-rook equivalence is the same as that a rook equivalence.
  - If $\lambda_i \geq i$, they came out with

$$R_n(F_\lambda, q) = \prod_{i=1}^{n} [\lambda_i - i + 1]_q, \qquad [m]_q := \frac{q^m - 1}{q - 1}. \tag{1}$$

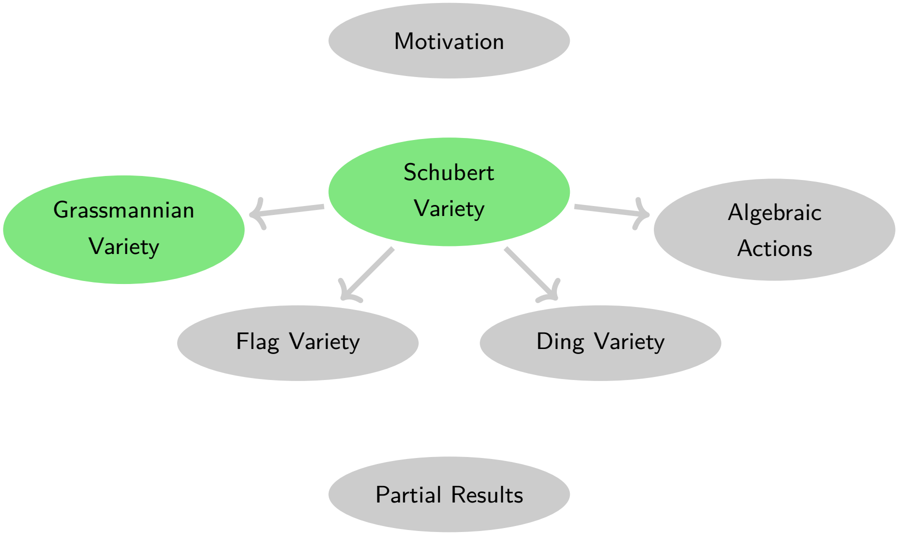
⇝ Homogeneous Spaces

- *K. Ding (2001)* depicted (1) as the Poincaré series for certain algebraic variety X_λ.
- X_λ is a smooth Schubert variety inside the partial flag variety X_{N^n} where N^n still stands for the board with n rows and N columns.
- The Schubert varieties popping up this way are those of the form X_w where w is a 312-avoiding permutation.
  - Likewise, the **fundamental cohomology class** $[X_w]$ is performed by a Schubert polynomial indexed by a **dominant or 123-avoiding** permutation.

Proposition (Can-D)

The number of 312-avoiding permutations of $\{1, \ldots n\}$ whose associated Schubert variety is a toric variety is 2^{n-1}.

Outline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Partial Results

Main Characters $\rightsquigarrow$ 15^{th} Hilbert's problem

- Let V be a vector space over $\mathbf{k}$. The Grassmannian **variety** is

$$\mathrm{Gr}(d,n) := \{W \subset V : W \text{ linear subspace and } \dim(W) = d\}.$$

e.g. $\mathrm{Gr}(2,4)$ i.e. $W = \mathrm{span}(w_1, w_2)$ and $\mathrm{span}(e_1, e_2, e_3, e_4) = \mathbf{k}^4$.

$$W \in \mathrm{Gr}(2,4) \iff W = \mathrm{span}\left\{\sum_{j=1}^{4} a_{1j} e_j, \sum_{j=1}^{4} a_{2j} e_j\right\} \in \mathrm{Gr}(2,4)$$

$$\iff \text{rows of } M_W \text{ are independent vectors in } \mathbf{k}^4$$

$$\iff \text{some } 2\times 2 \text{ minors of } M_W \text{ is } \textbf{NOT} \text{ zero}$$

$$\iff \underbrace{p_{j_1 j_2}(M_W)}_{\text{Plücker relations}} := \det[a_{p,j_q}]_{1\le p,q\le 2} \neq 0 \text{ for some } j_1 < j_2$$

$$\rightsquigarrow w_1 \wedge w_2 = \sum_{j_1<j_2} p_{j_1 j_2}(M_W) e_{j_1} \wedge e_{j_2}.$$

Proposition (Plücker embedding)

$$\begin{array}{rcl} \mathrm{Gr}(d,n) & \xrightarrow{\quad\psi\quad} & \mathbb{P}^{\binom{n}{d}-1} = \mathbb{P}\left(\bigwedge^k \mathbf{k}^n\right) \\ \mathrm{span}(v_1,\dots,v_d) & \longmapsto & [v_1 \wedge \cdots \wedge v_d] \end{array}.$$

- ψ *is injective.*
- $\operatorname{im}\mathrm{Gr}(d,n)$ *is closed in* $\mathbb{P}(\bigwedge^k \mathbf{k}^n)$.
  - $$\begin{array}{rcl} \mathbf{k}^n & \xrightarrow{\;h_\omega\;} & \Lambda^{d+1}\mathbf{k}^n \\ v & \longmapsto & v \wedge \omega \end{array}, \Rightarrow \dim(\operatorname{im} h_\omega) \geq n-d$$

e.g. $\mathrm{Gr}(2,4) \rightsquigarrow 3\times 3$ minors of $[h_\omega] \in \mathbf{Mat}_4 \rightsquigarrow 16$ cubic equations!

- $\mathrm{Gr}(d,n)$ can be covered by $\mathbb{A}^{d(n-d)}$ since

$$\mathbb{A}^{d(n-d)} \xrightarrow{\;\phi\;} V_I;\, C \longmapsto \mathrm{RowSpan}[I_d \mid C] \;.$$

  - $\mathrm{Gr}(d,n)$ is irreducible and $\dim(\mathrm{Gr}(d,n)) = d(n-d)$
  - $\mathrm{Gr}(2,4) = \mathbf{V}(a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23})$

Example: Schubert Cells

$$W := \operatorname{span}\left\{\begin{matrix} -e_2 - 3e_3 - e_4 + 6e_5 - 4e_6 + 5e_7 \\ e_2 + 3e_3 + 2e_4 - 7e_5 + 6e_6 - 5e_7 \\ 2e_4 - 2e_5 + 4e_6 - 2e_7 \end{matrix}\right\} \in \mathrm{Gr}(3,7)$$

$$M_W := \begin{pmatrix} 0 & -1 & -3 & -1 & 6 & -4 & 6 \\ 0 & 1 & 3 & 2 & -7 & 6 & -5 \\ 0 & 0 & 0 & 2 & -2 & 4 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1/2 & 1/2 & -1/4 \\ 1/2 & 1/2 & 1/4 \\ 3/2 & 5/2 & -7/4 \end{pmatrix} M_W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & 3 & 0 & -5 & 2 & 0 \end{pmatrix}$$

$$M'_W := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 1 & * & 0 & * & * & 0 \end{pmatrix} \rightsquigarrow \text{Schubert Cell}.$$

Recipe: cut out the $d \times d$ staircase from the upper left corner of the matrix, and let λ_i be the distance from the edge of the staircase to the 1 in row i.

Schubert Cell X_λ

For a partition λ contained in a rectangle $d(n-d)$ is the set of points of $\mathrm{Gr}(d,n)$ whose row echelon matrix has corresponding partition:

$$X_\lambda := \left\{ W \in \mathrm{Gr}(d,n) \mid \dim(W \cap \langle e_1, \ldots, e_r \rangle) = i, \begin{array}{c} n-d+i-\lambda_i \\ \leq r \leq \\ n-d+i-\lambda_{i+1} \end{array} \right\}$$

- The numbers $n-d+i-\lambda_i$ are the *positions* of the $1'$s in the matrix counted from the right.
- Since each $*$ can be any complex number, we have $X_\lambda = \mathbb{A}^{d(n-k)-|\lambda|}$ as a set, and so $\dim X_\lambda = d(n-d) - |\lambda|$.
    - In particular, $\dim \mathrm{Gr}(d,n) = d(n-d)$.
- *The d-subsets of $[n]$ is in bijection with partitions whose Ferrer diagram is contained in the $d(n-d)$ rectangle.*

Closed subsets: Schubert Variety

The Schubert Variety is the closure of X_λ i.e.,

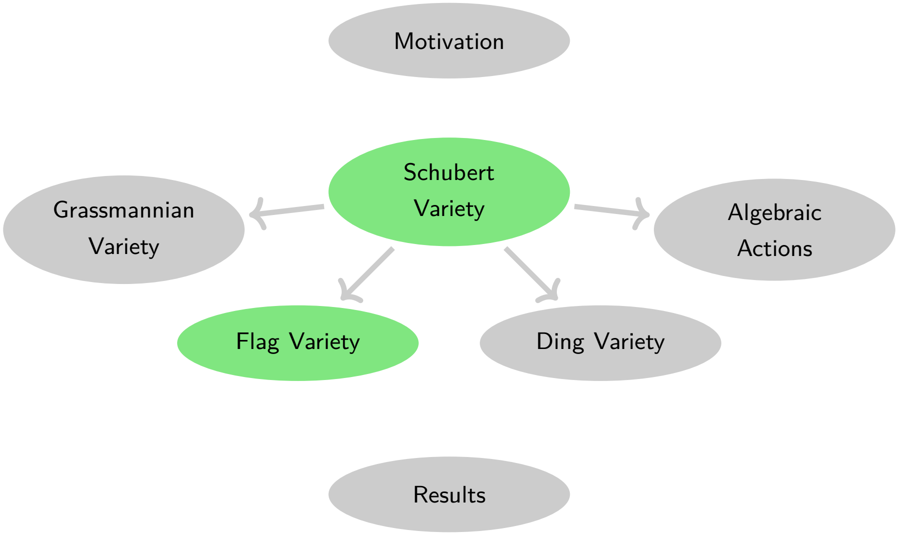
$$\overline{X_\lambda} := \{W \in \mathrm{Gr}(d,n) \mid \dim(V \cap \langle e_1, ..., e_{n-d+1-\lambda_i}\rangle) \geq i\}.$$

e.g. In $\mathrm{Gr}(2,4)$,

$$\overline{X}_{\{1,3\}} = \overline{X}_{\ydiagram{2,1}} = \overline{\left\{ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & * & 0 \end{pmatrix} \right\}} = X_{(2,2)} \sqcup X_{(2,1)}.$$

- Every Schubert variety is a disjoint union of Schubert cells.
- How many lines intersect four lines in $\mathbb{R}^3$?
  - $\rightsquigarrow$Given a line in $\mathbb{R}^3$, the family of lines intersecting it can be seen $\mathrm{Gr}(2,4)$ as the Schubert variety $\overline{X}_{\{2,4\}}$
- How many points $W \in \mathrm{Gr}(2,4)$ are in the intersection of 4 copies of the Schubert variety $\overline{X}_{(2,1)}$ each w.r.t a different basis?

Outline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Flag Variety

- Let V be a vector space over $\mathbf{k}$ with $\dim_{\mathbf{k}} V = n$ and let $1 \leq \ell_1 < \cdots < \ell_d \leq n$. A flag of type $(\ell_1, \dots, \ell_d)$ in V is a sequence of linear subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_d \subseteq V$, where $\dim_{\mathbf{k}}(V_i) = \ell_i$.
  - A complete flag is a flag of type $(1, 2, \dots, n)$.
- The flag variety $\mathcal{F}_{\ell_1,\dots,\ell_d}(V)$ parametrizes flags in V i.e., it is the set

$$\mathcal{F}_{\ell_1,\dots,\ell_d}(V) := \{(V_1, \dots, V_d) \in \mathrm{Gr}(\ell_1, V) \times \cdots \times \mathrm{Gr}(\ell_d, V) \mid V_1 \subseteq \cdots \subseteq V_d\}.$$

In particular, the complete flag variety $\mathcal{F}_{1,\dots,n}(V) := \mathcal{F}_\bullet(V)$ parametrizes complete flags in V.

Proposition

The subset $\mathcal{F}_{\ell_1,\dots,\ell_d}(V)$ *of* $\mathrm{Gr}(\ell_1, V) \times \cdots \times \mathrm{Gr}(\ell_d, V)$ *is closed and hence* $\mathcal{F}_{\ell_1,\dots,\ell_d}(V)$ *is a projective variety.*

- *For every* $(\ell_1, \dots, \ell_d)$*, the flag variety is irreducible.*
- *Its dimension is given by*

$$\sum_{i=1}^{d} \ell_i(\ell_{i+1} - \ell_i), \qquad \ell_{d+1} = n.$$

  - *For* $\mathcal{F}_\bullet(V)$*, its dimension is* $\dfrac{n(n-1)}{2}$.

Example: more tools from combinatorics

- $\mathcal{F}_\bullet = \text{span}\{2e_2 + 3e_3, e_1 + e_2 + 4e_3, e_1 + 2e_2 - 3e_3\}$

$$\leftrightsquigarrow \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/17 & 2/17 & -2/17 \end{pmatrix} \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 4 \\ 1 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 3/2 \\ 1 & 0 & 5/2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{span}\{e_2 + 3/2e_3, e_1 + 5/2e_3, e_3\} \rightsquigarrow \text{canonical form}.$$

  - The canonical form determines a **permutation matrix:** the *position* of the leading 1's. This permutation dictates the **position** of the flag $\mathcal{F}_\bullet$ w.r.t the reference flag $E_\bullet := \langle e_1, e_2, e_3 \rangle$.

- Ways of seeing permutations:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = 231$$

Schubert Cells in $\mathcal{F}_n(\mathbf{k})$

Let $w \in S_n$ be a permutation. The Schubert Cell of w is defined by

$$X_w := \{W_\bullet \in \mathcal{F}_n : \dim(W_p \cap E_q) = \#\{i \le p : w(i) \le q\} \text{for all } p, q\}$$

where E_q is the standard flag generated by the unit vectors e_{n+1-i}.

- The $\dim(X_w)$ is the $\#$ of $*$'s in its matrix.
  - The max $\#$ of $*$'s occurs when $w_0 = n(n-1)\cdots 321 \rightsquigarrow \dim(X_{w_0}) = n(n-1)/2$.
- For any $w \in S_n$, $\text{inv}(w) = \#\{(i,j) : i < j \text{ and } w(i) > w(j)\}$.
  - The $\#$ of $*$'s in X_w is the inversion number $\text{inv}(w)$
- Define $s_1, \ldots, s_{n-1} \in S_n$ to be the *adjacent transpositions* in S_n. Then the length of w, $\ell(w)$ is the smallest number k for which there exists a decomposition $w = s_{i_1} \cdots s_{i_k}$.
  - $\ell(w) = \text{inv}(w)$.

Keeping the example...

$$\mathcal{F}_\bullet := \begin{pmatrix} 0 & 1 & 3/2 \\ 1 & 0 & 5/2 \\ 0 & 0 & 1 \end{pmatrix} \in X_{213} = \left\{ \begin{pmatrix} 0 & 1 & * \\ 1 & 0 & * \\ 0 & 0 & 1 \end{pmatrix} : * \in \mathbb{C} \right\}$$

- $\dim(X_{231}) = 2 = \mathrm{inv}(231)$.
- $X_w = w \cdot B$ is a B-**orbit** using the **right** B **action** i.e.,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_{1,3} & 0 & 0 \\ b_{2,3} & b_{2,2} & 0 \\ b_{3,3} & b_{3,2} & b_{3,1} \end{pmatrix} = \begin{pmatrix} b_{2,3} & b_{2,2} & 0 \\ b_{1,3} & 0 & 0 \\ b_{3,3} & b_{3,2} & b_{3,1} \end{pmatrix}$$

Schubert Variety

Let $w \in S_n$ be a permutation. The Schubert Variety of w is defined by

$$\overline{X_w} := \{W_\bullet \in \mathcal{F}_n : \dim(W_p \cap E_q) \geq \#\{i \leq p : w(i) \leq q\} \text{for all } p, q\}$$

- The closure relation on Schubert varieties defines a cool partial order

$$\overline{X_w} = \bigsqcup_{v \leq w} X_v \rightsquigarrow \text{Bruhat order}$$

where $v \leq w \iff$ for every representation of w as a product of $\ell(w)$ transpositions s_i, one can remove $\ell(w) - \ell(v)$ of the transpositions to obtain a representation of v as a subword in the same relative order.

  - $w = 45132 = s_2 s_3 s_2 s_1 s_4 s_3 s_2$ and this contains $s_3 s_2 s_3 = 14325$ as a nonconsecutive subword and so $14325 \leq 45132$.

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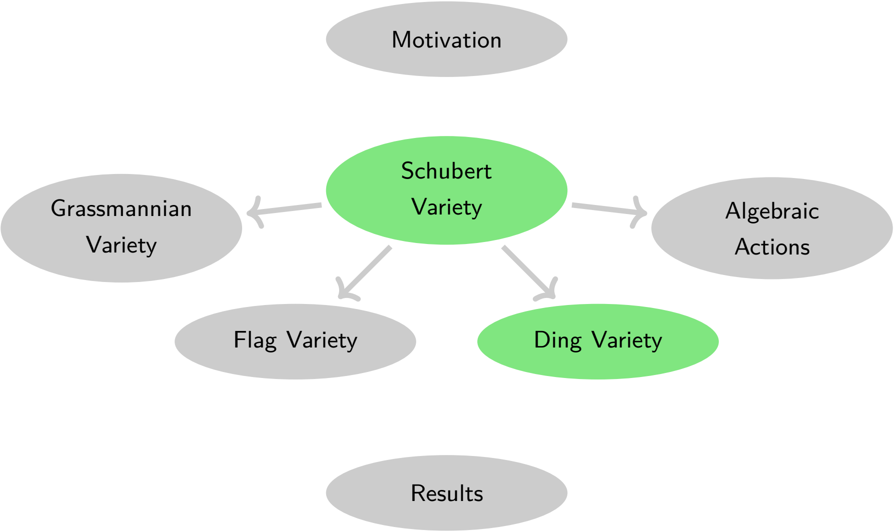
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Ding's Variety

Let $\lambda = (0 \leq \lambda_1 \leq \cdots \leq \lambda_d)$ be a partition.

$$D_\lambda := \{V_\bullet \in \mathcal{F}_{(1,\ldots,d)} \mid V_i \subset \mathbb{C}^{\lambda_i}\} \rightsquigarrow \text{Ding variety}$$

$$X_w := \left\{V_\bullet \in \mathcal{F}_{(1,\ldots,d)} \mid \dim(V_i \cap E_j) \geq \#\{k \leq i : w_k \leq j\}\right\}$$

D_λ coincides with $X_w \subset \mathcal{F}_{(1,\ldots,d)}(V)$, where w is the unique permutation given by $w_i = \max(\{1, \ldots, \lambda_i\} \setminus \{w_1, \ldots, w_{i-1}\})$.

- If $d = n$, then w identifies the maximal rook placement on the Ferrers board B_λ.
  - For each i, place a rook in row i and column w_i, where w_i is the rightmost square in row i whose column doesn't contain a rook.
- Permutation w obtained this way are exactly those which are 312-avoiding.
  - There do not exist i, j, k for which $i < j < k$ and $w_i > w_k > w_j$.

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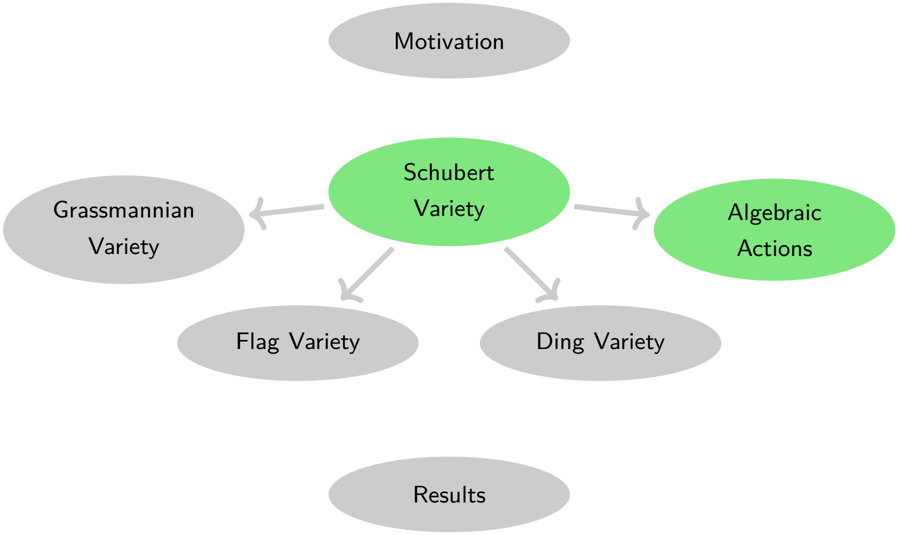
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Actions

- A G-variety is an affine variety X endowed with an action of the algebraic group G, $\begin{array}{rcl} G \times X & \xrightarrow{\alpha} & X \\ (g,x) & \longmapsto & g \cdot x \end{array}$ which is a morphism.
- $x \in X$ is called fixed point if $gx = x$ for all $g \in G$.
  - $X^G := \{x \in X : x \text{ fixed point}\}$
- For $x \in X$, the orbit of x is $Gx := \{gx : g \in G\} \subset X$.
  - The orbit map is $\alpha_x : G \to X; g \mapsto gx$.
- The stabilizer of x is $\mathrm{Stab}_G(x) = G_x := \{g \in G : gx = x\}$.
  - For any $Y \subset X$, we define $\mathrm{Stab}_G(Y) := \{g \in G : gy = y, \forall y \in Y\}$.
- For two G-varieties X, Y a morphism $\varphi : X \to Y$ is said to be G-equivariant whether $\varphi(gx) = g\varphi(x)$ for all $g \in G$ and $x \in X$.

Homogeneous Spaces

Proposition

Let X be a G-variety.

- *X^G is a closed subset in X.*
- *G_x and $\mathrm{Stab}_G(Y)$ are closed subgroups of G.*
- *For any $x \in X$ the orbit Gx is open in its closure $\overline{Gx}$.*
- *If $Y \subset X$ is closed, the normalizer $N_G(Y)$ is closed subgroup of G.*

- A variety X is homogeneous whether it is equipped with a transitive action of an alg. group G.
  - A homogeneous space is (X, x) where X is homogeneous and $x \in X$ is called base point.
- Since $\alpha_x^{-1}(hx) = hG_x \implies \dim Gx = \dim \overline{Gx} = \dim G - \dim G_x$
  - $\overline{Gx}$ contains a closed orbit.

Example

$$\begin{array}{ccc} \mathrm{GL}_n \times \mathrm{Gr}(d,n) & \xrightarrow{\alpha_1} & \mathrm{Gr}(d,n) \\ & & \downarrow \psi \\ \mathrm{GL} \times \mathbb{P}(\bigwedge \mathbb{C}^n) & \xrightarrow{\alpha_2} & \mathbb{P}(\bigwedge \mathbb{C}^n) \end{array}$$

GL_n-equivariant, it is a unique GL_n-orbit.

$$\mathrm{Stab}_{\mathrm{GL}_n}(\langle e_1, \ldots, e_d \rangle) := P = \left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,d} & a_{1,d+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{d,1} & \cdots & a_{d,d} & a_{d,d+1} & \cdots & a_{d,n} \\ 0 & \cdots & 0 & a_{d+1,d+1} & \cdots & a_{d+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n,d+1} & \cdots & a_{n,n} \end{pmatrix} \right\}$$

- $P \rightsquigarrow$ maximal parabolic subgroup of GL_n, and $\mathrm{Gr}(d,n)$ is an homogeneous space GL_n / P.
  - $\dim(\mathrm{Gr}(d,n)) = \dim \mathrm{GL}_n - \dim P = d(n-d)$

Pick $E_I = \langle e_{i_1}, ..., e_{i_d} \rangle \in \mathrm{Gr}(d,n)$ where $1 \le i_1 < \cdots < i_d \le n$.

- The E_I are the T-fixed points in $\mathrm{Gr}(d,n)$, where

$$T_n = \left\{ \begin{pmatrix} a_{1,1} & 0 & \cdots & 0 \\ 0 & a_{2,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{pmatrix} \right\} \rightsquigarrow \text{maximal torus of } \mathrm{GL}_n$$

- $\mathrm{Gr}(d,n)$ is the disjoint union if the orbits BE_I, where

$$B_n := \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ 0 & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n,n} \end{pmatrix} \right\} \rightsquigarrow \text{Borel subgroup of } \mathrm{GL}_n$$

- The Schubert Cells in $\mathrm{Gr}(d,n)$ are the orbits $X_I := BE_I$ and the closure in $\mathrm{Gr}(d,n)$ of X_I is called Schubert variety $\overline{X_I}$.

Examples

$$\begin{aligned} \mathrm{GL}_n \times \mathcal{F}(\mathbb{C}^n) &\xrightarrow{\alpha} \mathcal{F}(\mathbb{C}^n) \\ (B, F) &\longmapsto BF \end{aligned}$$ is homogeneous.

$$\mathrm{Stab}_{\mathrm{GL}_n}(E_1, \ldots, E_d) := \overbrace{P_{\ell_1,\ldots,\ell_d}(\mathbb{C})}^{\text{Parabolic}} = \begin{pmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & A_d \end{pmatrix},$$

$$A_i \in \mathrm{GL}_{d_i - d_{i-1}}$$

Hence,

$$\begin{aligned} \mathrm{GL}_n / P_{\ell_1,\ldots,\ell_d}(\mathbb{C}) &\longrightarrow \mathcal{F}_{\ell_1,\ldots,\ell_d}(V) \\ gP_{\ell_1,\ldots,\ell_d}(\mathbb{C}) &\longmapsto (gV_1, \ldots, gV_d). \end{aligned}$$

- The set of T-fixed points is identified with S_n. Namely, each $w \in S_n$ corresponds to a coordinate flag

$$\{0\} \subset \langle e_{w(1)} \rangle \subset \langle e_{w(1)}, e_{w(2)} \rangle \subset \cdots \subset V_n = \mathbb{C}^n,$$

denotes as $E_w := wE$
- $S_n \cong N_G(T)/T \rightsquigarrow$ Weyl group , $\mathcal{F}_n$ is the disjoint union of the orbits $X_w := BF_w = UF_w$ where $w \in S_n$.
- When $\overline{X_w}$, we acquire

$$\overline{X_w} = \bigsqcup_{v \in W, v \leq w} X_v$$

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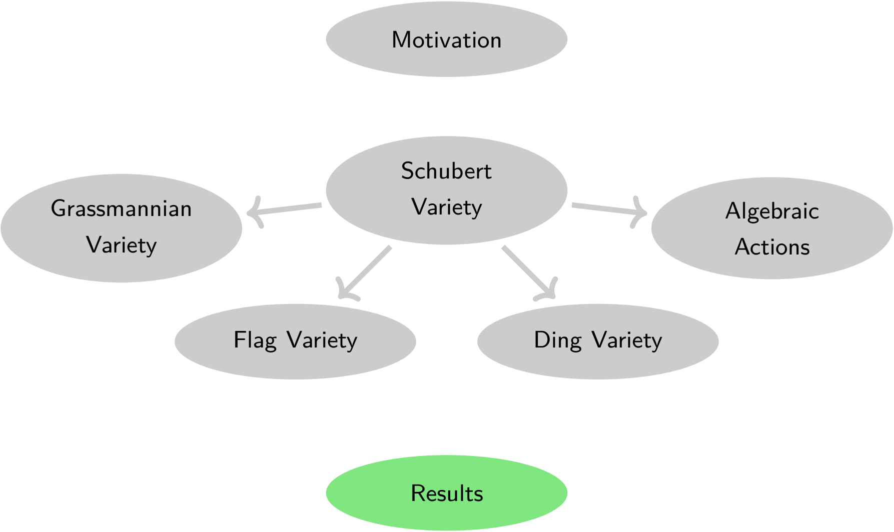
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Current work...

Theorem (Karuppuchamy)

A Schubert variety X_w is a toric variety if and only if the Weyl group element w is a product of distinct simple reflections.

Theorem (Lakshmibai-Sandhya)

X_w is non-singular if and only if w has no subsequence with the same relative order as 3412 and 4231.

Proposition (Can-D)

The number of 312-avoiding permutations of $\{1, ...n\}$ whose associated Schubert variety is a toric variety is 2^{n-1}.

Classification: Schubert variety of complexity 0

Theorem

The following are equivalent

(1) X_w *is a toric variety i.e., of complexity 0.*

(2) X_w *is a smooth toric variety.*

(3) w *avoids the patterns* 321 *and* 3412.

(4) *A reduced decomposition of* w *consists of distinct letters.*

(5) X_w *is isomorphic to a* ***Bott-Samelson variety****.*

(6) *The* ***Bruhat interval*** $[e, w]$ *is isomorphic to* $\mathfrak{B}_{\ell(w)}$ *the* ***Boolean*** *algebra of rank* $\ell(w)$*.*

(7) *The* ***Bruhat interval polytope*** $Q_{e,w}$ *is combinatorially equivalent to the* $\ell(w)$*-dimensional cube.*

Classification: Schubert variety of complexity 1

Theorem

For a permutation $w \in S_n$, the following are equivalent

(1) *X_w is smooth and of complexity 1.*

(2) *w contains the pattern 321 exactly once and avoids the pattern 3412.*

(3) *There exists a reduced decomposition of w containing $s_i s_{i+1} s_i$ as a factor and no other repetitions.*

(4) *X_w is isomorphic to a **flag Bott-Samelson** variety $Z_{(w_1,\ldots,w_r)}$ such that $r = \ell(w) - 2$, $w_k = s_j s_{j+1} s_j$ for some $1 \leq k \leq r$, $w_i = s_{j_i}$ for $i \neq k$, and $j_1, \ldots, j_{k-1}, j_{k+1}, \ldots, j_r, j, j+1$ are pairwise distinct.*

(5) *The **Bruhat interval** $[e, w]$ is isomorphic to $S_n \times \mathfrak{B}_{\ell(w)-3}$*

(6) *The **Bruhat interval polytope** $Q_{e,w}$ is combinatorially equivalent to the product of the hexagon and the cube $I^{\ell(w)-3}$.*

Thank You/Gracias!

“Combinatorics is the nanotechnology of mathematics”
Sara Billey

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